KEY

TO

PROFESSOR YOUNG'S ALGEBRA.

BY

W. H. SPILLER.

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TO

J. R. YOUNG, Esq.,

PROFESSOR OF MATHEMATICS IN THE ROYAL COLLEGE OF BELFAST; HONORARY ASSOCIATE OF THE VERULAM PHILOSOPHICAL SOCIETY OF LONDON, ETC.

My dear Sir,

In dedicating the present work to you, it is not my intention to offer any eulogium on your talents, for these have long since raised you among mathematicians to an elevation far above the reach of my humble praises. To the excellent qualities of your heart, however, which have so justly rendered you an object of esteem and attachment to all your friends, I may be permitted to bear record; and, at the same time, to assure you how truly I feel, and how highly I appreciate, the sincere and uniform regard you have always manifested towards me, and which I rejoice to find has not at all been weakened by the distance which now separates us.

But, apart from the influence of such considerations as these, I feel it to be befitting that these pages should be

submitted to you: for, to whom could I so appropriately inscribe them as to the Author of the work which they attempt to illustrate—to him from whom I first imbibed the love of those pursuits which have proved to me the source of some of the purest enjoyments I have ever experienced?

With unfeigned gratitude and respect, believe me,

My dear Sir,

Ever faithfully your's,

W. H. SPILLER.

December, 1835.

PREFACE.

As this Key is chiefly intended for the assistance of *beginners*, I have paid the utmost attention to every particular which might in any way facilitate their progress; and, in order to reduce everything to the comprehension of the pupils, I have simplified the solutions as much as possible, and worked out the problems in full, without omitting any steps which might at all assist them; indeed, I have even ventured, when any very useful transformations have presented themselves, to insert one or two steps more than are actually necessary, in consequence of the great advantage the student will derive from it (this will be particularly observed in Ex. 22, Quad. Eq. page 72); so that I may say I have not always been desirous of giving the *shortest* solutions, but those which occurred to me as being the *most intelligible* to a beginner; and it may be remarked here, that after the student has gone over a certain number of the equations, and *thoroughly understands them*, he would derive great benefit from exercising himself in solving some of the questions anew, paying somewhat more attention to brevity. Experience and practice, however, will soon point out the best and easiest methods.*

* The beginner who has not the advantage of a master, would do well to read Lacroix's Elements of Algebra (lately translated) in conjunction with this work, as it abounds in copious illustrations explained in the most familiar manner.

That the pupil may keep in view the reason of every step he advances, I have given every assistance, by means of notes and marginal references; these latter occur chiefly in equations, where in any such expression as this ($x = \sqrt{x^2}$), within a parenthesis (see page 72 among the marginal references on the left hand), the student will observe that it is merely explanatory of the step opposite to which it stands; for instance, in the case just alluded to, in the second step we have the equation $\frac{x^2}{\sqrt{2x^4 - 3x^3}} = 2x$, which, by taking $\sqrt{x^2}$ from each term of the denominator, and putting it as a factor, may be written $\frac{x^2}{\sqrt{x^2}\sqrt{2x^2-3x}} = 2x$; and, as the marginal reference ($x = \sqrt{x^2}$) indicates, x is equal to x^2 under the radical sign $\sqrt{}$; hence, by substituting x for $\sqrt{x^2}$, the expression becomes $\frac{x^2}{x\sqrt{2x^2 - 3x}} = 2x$. The learner, however, when he shall have arrived at that part of the subject, will have no difficulty in understanding these matters. Sometimes these marginal references, for want of room on the left-hand side of the step they illustrate, are put on the right.

It will be observed that, in the various solutions, I have uniformly adopted the methods given in the Algebra.

In passing through the present edition (1834) of Mr. Young's Algebra, I have met with several errors of the press, a list of which will be inserted in that work.

W. H. S.

Dec. 1835.

KEY

TO

YOUNG'S ALGEBRA.

ADDITION.

Case I.—Page 5.

When the quantities are like, but have unlike signs.

Ex. 6. Ans. $16a^2 + 9\,bx$.

Ex. 7. Ans. $16\sqrt{y} - 4\,(a + b)$.

Ex. 8. Ans. $11a\,(a + b) + 16\sqrt{a - x}$.

Ex. 9.* Ans. $18x^{\frac{1}{2}}y + 3xy^{\frac{1}{2}} + 1$.

Ex. 10. Ans. $-25\,(x + y)^{\frac{1}{2}} + 12\,(xyz)^{\frac{1}{2}}$.

Ex. 11. Ans. 0 .

Ex. 12. Ans. $5\,(ax + by + cz)^{\frac{1}{4}} - 4\,(a - b)$.

Case II.—Page 8.

When both quantities and signs are unlike, or some like and others unlike.

Ex. 3. Ans. $3\frac{1}{2}\sqrt{x} + 7\frac{1}{2}\,ax - \sqrt{b - x} + 10\,xy - 19$.

Ex. 4. Ans. $2ab - x^2y + 5\,xy + 4xy^{\frac{1}{2}} + 60$.

Ex. 5. Ans. $8\,(x^2 + y^2)^{\frac{1}{2}} + 3\,(x^2 - y^2)^{\frac{1}{2}} - 9xy$.

Ex. 6. Ans. $7\sqrt[4]{xy} - 4a\sqrt{x} + 20x^2 - 9\sqrt{xy} + 3y^2 - 24$.

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* In adding up the second column of this Example, the student must not forget that $\sqrt{y}$ and $y^{\frac{1}{2}}$ are the same. See Note, page 7, of the Algebra.

When the coefficients are *literal* instead of *numeral*, &c.

Ex. 3. Ans. $(b + d + 1\frac{1}{2})\,x^2 + (ad + ce - m - n)\,z$.

Ex. 4. Ans. $ax + (c + 1)\sqrt{x} + (am + b + 1)\,y + (d - 1)\,z$.

Ex. 5. Ans. $(5a - d - e + 1)\sqrt{x^2 - y^2} + (b + c + f + 1)\,(x^2 + y^2)^{\frac{1}{2}}$.

Ex. 6. Ans. $(2a + 2b + 3c - 2d + e - 2n)\sqrt{x} + (12a + 4n + 2)\,y^{\frac{1}{2}}$.

SUBTRACTION.

Page 10.

Ex. 3. Ans. $a\,(a - y) + 11by - 3a^3$.

Ex. 4. Ans. $-\,2x^2 + 2\sqrt{x + y} - 10c - 1$.

Ex. 5. Ans. $9abx - 8xy + 3xz + 19$.

Ex. 6. Ans. $-\,3\sqrt{x^2 - y^2} + \sqrt{a + x} + 4$.

Ex. 7. Ans. $3x\,(x + y)^{\frac{1}{4}} + 14axy - 9abc$.

Examples of quantities with literal coefficients.

Ex. 2. Ans. $(p - m)\,xy + (p + 1)\,(qxz) + (n - r)\,z^2 + s - a$.

Ex. 3. Ans. $(a - 1)\,(x - y)^{\frac{1}{2}} + 2\,bxy - a\,(a + x)^2$.

Ex. 4. Ans. $2b\,(x + y) - 2c\,(x - y) + m + n$.

Ex. 5. Ans. $(2a - b)\,xy - (3p - 2q)\sqrt{x + y} + 3x^2$.

MULTIPLICATION.

CASE I.—Page 14.

When both multiplicand and multiplier are simple quantities.

Ex. 3. Ans. $-48\,ax^{\frac{1}{2}}\,y$.

Ex. 4. Ans. $16x^4y^5$.

Ex. 5. Ans. $18a^3bx^2y^2$.

Ex. 6. Ans. $-104a^4b^3x^4y^6$.

Ex. 7. Ans. $3x^6y^6z^6$.

Ex. 8. Ans. $3c^3x^2y^4z^7$.

CASE II.—Page 14.

When the multiplicand is a compound quantity, and the multiplier a simple quantity.

Ex. 3. Ans. $25abxy + 15axy - 10xy$.

Ex. 4. Ans. $-62xy^2\sqrt{b} + 8\sqrt{bx} - 2a\sqrt{b}$.

Ex. 5. Ans. $36ax^4y + 6ax^2y^2 + 3ax^2y$.

Ex. 6. Ans. $12abx^2y^2 + 9cxy^3 - 3abcxy^2$.

Ex. 7. Ans. $-21x^5y^4 + 28x^4y^3 - 7bx^4y$.

Ex. 8. Ans. $-40a^2x^2y^3 + 4ax^2y - 2a^2xy^4$.

Ex. 9. Ans. $-2a^2x^2\sqrt{z} + 4a^3x^4 + 3a^2x^3y^2$.

DIVISION.

Case I.—Page 17.

When both dividend and divisor are simple quantities.

Ex. 4. $9a^3x^4 \div 3ax^2$, or $\dfrac{9a^3x^4}{3ax^2} = 3a^2x^2$.

Ex. 5. $\dfrac{26ax^2y^2}{-2xy} = -13axy$.

Ex. 6. $\dfrac{-15\,b^3xy^5}{-bxy^2} = 15b^2y^3$.

Ex. 7. $\dfrac{28c^4z^6}{-7c^3z^6} = -4c$.

Ex. 8. $\dfrac{-18a^2b^3y^7z^4}{-aby^3z} = 18ab^2y^4z^3$.

Case II.—Page 18.

When the dividend is a compound quantity, and the divisor a simple quantity.

Ex. 3. $\dfrac{24a^2x^4 + 6a^3x^3 - 3a^4x^2 + 12ax}{-3ax} = -8ax^3 - 2a^2x^2 + a^3x - 4$.

Ex. 4. $\dfrac{ax^n + ax^{n+1} + ax^{n+2} + ax^{n+3} + \&c.}{x^n} = a + ax + ax^2 + ax^3 + \&c.$

Ex. 5. $\dfrac{6(x+y)^3 - 8(x+y)^2 + 4a^2(x+y)}{2(x+y)} = 3(x+y)^2 - 4(x+y) + 2a^2$.

Ex. 6. $\dfrac{ax^{m-1} + bx^{m+1} - cx^{m-3} + dx^5}{x^{m-6}} = ax^5 + bx^7 - cx^3 + dx^{11-m}$.

CASE III.—Page 20.

When both dividend and divisor are compound quantities.

Ex. 4. $a + x)\ a^5 + x^5\ (a^4 - a^3x + a^2x^2 - ax^3 + x^4$

$$
\begin{array}{l}
a^5 + a^4x \\
\hline
-a^4x + x^5 \\
-a^4x - a^3x^2 \\
\hline
a^3x^2 + x^5 \\
a^3x^2 + a^2x^3 \\
\hline
-a^2x^3 + x^5 \\
-a^2x^3 - ax^4 \\
\hline
ax^4 + x^5 \\
ax^4 + x^5 \\
\hline
\ast \quad \ast \\
\hline
\end{array}
$$

Ex. 5. $a - x)\ a^5 - x^5\ (a^4 + a^3x + a^2x^2 + ax^3 + x^4$

$$
\begin{array}{l}
a^5 - a^4x \\
\hline
a^4x - x^5 \\
a^4x - a^3x^2 \\
\hline
a^3x^2 - x^5 \\
a^3x^2 - a^2x^3 \\
\hline
a^2x^3 - x^5 \\
a^2x^3 - ax^4 \\
\hline
ax^4 - x^5 \\
ax^4 - x^5 \\
\hline
\ast \quad \ast \\
\hline
\end{array}
$$

DIVISION.

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Ex. 8. $\dfrac{-18a^2b^3y^7z^4}{-aby^3z} = 18ab^2y^4z^3$.

CASE II.—Page 18.

When the dividend is a compound quantity, and the divisor a simple quantity.

Ex. 3. $\dfrac{24a^2x^4 + 6a^3x^3 - 3a^4x^2 + 12ax}{-3ax} = -8ax^3 - 2a^2x^2 + a^3x - 4$.

Ex. 4. $\dfrac{ax^n + ax^{n+1} + ax^{n+2} + ax^{n+3} + \&c.}{x^n} = a + ax + ax^2 + ax^3 + \&c.$

Ex. 5. $\dfrac{6(x+y)^3 - 8\,(x+y)^2 + 4a^2(x+y)}{2\,(x+y)} = 3\,(x+y)^2 - 4\,(x+y) + 2a^2$.

Ex. 6. $\dfrac{ax^{m-1} + bx^{m+1} - cx^{m-3} + dx^5}{x^{m-6}} = ax^5 + bx^7 - cx^3 + dx^{11-m}$.

Case III.—Page 20.

When both dividend and divisor are compound quantities.

Ex. 4.

$$
\begin{array}{l}
a+x)\ a^5+x^5\ (a^4-a^3x+a^2x^2-ax^3+x^4 \\
\qquad a^5+a^4x \\
\hline
\qquad -a^4x+x^5 \\
\qquad -a^4x-a^3x^2 \\
\hline
\qquad\qquad a^3x^2+x^5 \\
\qquad\qquad a^3x^2+a^2x^3 \\
\hline
\qquad\qquad\quad -a^2x^3+x^5 \\
\qquad\qquad\quad -a^2x^3-ax^4 \\
\hline
\qquad\qquad\qquad ax^4+x^5 \\
\qquad\qquad\qquad ax^4+x^5 \\
\hline
\qquad\qquad\qquad \cdot \quad \cdot \\
\hline
\end{array}
$$

Ex. 5.

$$
\begin{array}{l}
a-x)\ a^5-x^5\ (a^4+a^3x+a^2x^2+ax^3+x^4 \\
\qquad a^5-a^4x \\
\hline
\qquad a^4x-x^5 \\
\qquad a^4x-a^3x^2 \\
\hline
\qquad\qquad a^3x^2-x^5 \\
\qquad\qquad a^3x^2-a^2x^3 \\
\hline
\qquad\qquad\quad a^2x^3-x^5 \\
\qquad\qquad\quad a^2x^3-ax^4 \\
\hline
\qquad\qquad\qquad ax^4-x^5 \\
\qquad\qquad\qquad ax^4-x^5 \\
\hline
\qquad\qquad\qquad \cdot \quad \cdot \\
\hline
\end{array}
$$

Ex. 6. $\tfrac{1}{2}x + \tfrac{1}{2})\ x^3 + \tfrac{3}{4}x^2 + \tfrac{3}{4}x + 1\ (2x^2 - \tfrac{1}{2}x + 2$

$$\begin{array}{l}
x^3 + x^2 \\
\hline
\quad -\tfrac{1}{4}x^2 + \tfrac{3}{4}x \\
\quad -\tfrac{1}{4}x^2 - \tfrac{1}{4}x \\
\hline
\qquad\qquad x + 1 \\
\qquad\qquad x + 1 \\
\hline
\qquad\qquad * \quad * \\
\hline
\end{array}$$

Ex. 7. $1 - x)\ 1\ (1 + x + x^2 + x^3 + x^4 + x^5 + \text{\&c.}$

$$\begin{array}{l}
1 - x \\
\hline
\quad x \\
\quad x - x^2 \\
\hline
\qquad x^2 \\
\qquad x^2 - x^3 \\
\hline
\qquad\quad x^3 \\
\qquad\quad x^3 - x^4 \\
\hline
\qquad\qquad x^4 \\
\qquad\qquad x^4 - x^5 \\
\hline
\qquad\qquad\quad x^5 \\
\qquad\qquad\quad x^5 - x^6 \\
\hline
\qquad\qquad\qquad x^6\ \text{\&c.}
\end{array}$$

Ex. 8. $x^3+x^2y+xy^2+y^3)\ x^4-y^4\ (x-y$

$$\begin{array}{l}
x^4+x^3y+x^2y^2+xy^3 \\ \hline
\quad -x^3y-x^2y^2-xy^3-y^4 \\
\quad -x^3y-x^2y^2-xy^3-y^4 \\ \hline
\quad \ast \quad \ast \quad \ast \quad \ast \\ \hline
\end{array}$$

Ex. 9. $y^m+x^m)\ y^{m+1}+yx^m-y^mx-x^{m+1}\ \ (y-x$

$$\begin{array}{l}
y^{m+1}+yx^m \\ \hline
\qquad\qquad -y^mx-x^{m+1} \\
\qquad\qquad -y^mx-x^{m+1} \\ \hline
\qquad\qquad \ast \quad \ast \\ \hline
\end{array}$$

Ex. 10. $1+x)\ a-bx+cx^2-dx^3+\&c.\ \ \left(a-\begin{array}{r}a\\-b\end{array}\right|x+\begin{array}{r}a\\+b\\+c\end{array}\Bigg|x^2-\&c.$

$$\begin{array}{l}
a+ax \\ \hline
\left.\begin{array}{r}-a\\-b\end{array}\right|x+cx^2 \\
\left.\begin{array}{r}-a\\-b\end{array}\right|\left.\begin{array}{r}x-a\\-b\end{array}\right|x^2 \\ \hline
\qquad\qquad \left.\begin{array}{r}a\\b\\c\end{array}\right|x^2-dx^3 \\
\qquad\qquad \left.\begin{array}{r}a\\b\\c\end{array}\right|\left.\begin{array}{r}x^2+a\\+b\\+c\end{array}\right|x^3 \\ \hline
\qquad\qquad\qquad \left.\begin{array}{r}-a\\-b\\-c\\-d\end{array}\right|x^3\ \&c.
\end{array}$$

ALGEBRAIC FRACTIONS.

Page 22.

To reduce a mixed quantity to an improper fraction.

Ex. 4. $2ay + \frac{4x - ay + 2}{2xy} = \frac{4axy^2 + 4x - ay + 2}{2xy}$.

Ex. 5. $a - x - \frac{a^2 - ax}{x} = \frac{2ax - x^2 - a^2}{x}$.

Ex. 6. $ax - y - \frac{3ax - 4y - 2}{5} = \frac{2ax - y + 2}{5}$.

Ex. 7. $a + b - \frac{a^2 - b^2 - 3}{a - b} = \frac{a^2 - b^2 - a^2 + b^2 + 3}{a - b} = \frac{3}{a - b}$.

Ex. 8. $x^4 - x^3y + x^2y^2 - xy^3 + y^4 - \frac{1}{x+y} =$

$$\frac{x^5 + x^4y - x^4y - x^3y^2 + x^3y^2 + x^2y^3 - x^2y^3 - xy^4 + xy^4 + y^5 - 1}{x+y} =$$

$$\frac{x^5 + y^5 - 1}{x+y}.$$

To reduce an improper fraction to a whole, or mixed quantity.

Ex. 2. $\frac{2xy - a}{xy} = 2 - \frac{a}{xy}$.

Ex. 3. $\frac{x^2 - y^2 + 4}{x + y} = x - y + \frac{4}{x+y}$.

Ex. 4. $\frac{3(a^5 + b^5) - 3}{a + b} = \frac{3a^5 + 3b^5 - 3}{a + b} = 3a^4 - 3a^3b +$

$$3a^2b^2 - 3ab^3 + 3b^4 - \frac{3}{a+b}.$$

Ex. 5. $\frac{4axy^2 + 4x - ay + 2}{2xy} = 2ay + \frac{4x - ay + 2}{2xy}$.

Ex. 6. $\dfrac{x^3 - y^3 + x^2 - 2y^2}{x - y} = x^2 + xy + y^2 + x + y - \dfrac{y^2}{x - y}.$

To find the greatest common measure of the terms of a fraction.

Ex. 3.

$$
\begin{array}{lll}
2x^2 + 3ax + a^2) & 2ax^2 - a^2x - a^3 & (a \\
 & 2ax^2 + 3a^2x + a^3 & \\
\hline
 & \quad - 4a^2x - 2a^3 &
\end{array}
$$

then, $-4a^2x - 2a^3$

or, expunging $-2a^2$,

$$
\begin{array}{lll}
2x + a) & 2x^2 + 3ax + a^2 & (x + a \\
 & 2x^2 + ax & \\
\hline
 & \quad 2ax + a^2 & \\
 & \quad 2ax + a^2 & \\
\hline
 & \quad * \quad * &
\end{array}
$$

and, dividing both numerator and denominator by this common measure $(2x + a)$, we have the fraction in its lowest terms $\dfrac{ax - a^2}{x + a}$.

Ex. 4. $6ax - 8a$

or, dividing by 2,

$$
\begin{array}{lll}
3ax - 4a) & 6ax^3 + ax^2 - 12ax & (2x^2 + 3x \\
 & 6ax^3 - 8ax^2 & \\
\hline
 & \quad 9ax^2 - 12ax & \\
 & \quad 9ax^2 - 12ax & \\
\hline
 & \quad * \quad * &
\end{array}
$$

by dividing both terms by $3ax - 4a$, we find $\dfrac{2x^2 + 3x}{2}$ for the lowest terms of the fraction.

Ex. 5. $x^3 + y^3)\ x^4 - y^4\ (x$

$$\begin{array}{l} x^4 + xy^3 \\ \hline -xy^3 - y^4 \end{array}$$

then, $-xy^3 - y^4$
or, expunging $-y^3$,
$x + y)\ x^3 + y^3\ (x^2 - xy + y^2$

$$\begin{array}{l} x^3 + x^2y \\ \hline -x^2y + y^3 \\ -x^2y - xy^2 \\ \hline xy^2 + y^3 \\ xy^2 + y^3 \\ \hline * \quad * \\ \hline \end{array}$$

and dividing both terms of the fraction by $x + y$, gives $\dfrac{x^3 - x^2y + xy^2 - y^3}{x^2 - xy + y^2}$.

Ex. 6. $2x^3 - 16x - 6$
or, dividing by 2,
$x^3 - 8x - 3)\ 3x^3 - 24x - 9\ (3$

$$\begin{array}{l} 3x^3 - 24x - 9 \\ \hline * \quad * \quad * \\ \hline \end{array}$$

and, dividing both terms of the fraction by $x^3 - 8x - 3$, we find $\dfrac{3}{2}$ for its most simple form.

To reduce fractions to a common denominator.—Page 29.

Ex. 2.

$$\left.\begin{array}{l} 4 \times 4 \times x = 16x \\ ax \times 2a \times 4 = 8a^2x \\ ax \times x \times 3 = 3ax^2 \end{array}\right\} \text{new numerators}$$

$ax \times x \times 4 = 4ax^2$, the common denominator.

$\therefore$ the fractions are $\frac{16x}{4ax^2}$, $\frac{8a^2x}{4ax^2}$, and $\frac{3ax^2}{4ax^2}$; or $\frac{16}{4ax}$, $\frac{8a^2}{4ax}$, and $\frac{3ax}{4ax}$, by dividing by x.

Ex. 3. $\left.\begin{array}{l}(2x+1)\times 3=6x+3\\ (x+a)\times a=ax+a^2\end{array}\right\}$ new numerators,

$a\times 3=3a$, the common denominator;

$\therefore$ the fractions are $\frac{6x+3}{3a}$, and $\frac{ax+a^2}{3a}$.

Ex. 4. $\left.\begin{array}{l}2a\times a=2a^2\\ 2a\times 4=8a\end{array}\right\}$ new numerators,

$2a$, the common denominator;

$\therefore$ the fractions are $\frac{2a^2-a}{2a}$, $\frac{2a^2}{2a}$, and $\frac{8a}{2a}$.

Ex. 5. $\left.\begin{array}{l}(3x^2-2)\times(a+x)=3ax^2-2a+3x^3-2x\\ (2x^2-x+4)\times 4a=8ax^2-4ax+16a\end{array}\right\}$ new numerators,

$(a+x)\times 4a=4a^2+4ax$, the common denominator;

$\therefore$ the fractions are $\frac{3ax^2-2a+3x^3-2x}{4a^2+4ax}$, and $\frac{8ax^2-4ax+16a}{4a^2+4ax}$.

Ex. 6. $\left.\begin{array}{l}a\ (x-y)\\ b\ (x+y)\end{array}\right\}$ new numerators,

$(x+y)\ (x-y)=x^2-y^2$, the common denominator;

$\therefore$ the fractions are $\frac{a\,(x-y)}{x^2-y^2}$, $\frac{b\,(x+y)}{x^2-y^2}$, and $\frac{c}{x^2-y^2}$.

Ex. 7. By multiplying together the denominators of the first, third, and fourth fractions, we shall have the same denominator as the second;

$\therefore \left.\begin{array}{l}(a+x)\ (a-x)^2\\ x\ (a-x)^2\\ x\ (a+x)\end{array}\right\}$ new numerators,

$x\ (a+x)\ (a-x)=x\,(a^2-x^2)$, the common denominator;

$\therefore$ the fractions are $\dfrac{(a+x)(a-x)^2}{x(a^2-x^2)}$, $\dfrac{a+x}{x(a^2-x^2)}$, $\dfrac{x(a-x)^2}{x(a^2-x^2)}$, and

$$\frac{x(a+x)}{x(a^2-x^2)}.$$

ADDITION OF FRACTIONS.

Page 31.

Ex. 2. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{x^2+2xy+y^2}{x^2-y^2}+\dfrac{x^2-2xy+y^2}{x^2-y^2}=\dfrac{2x^2+2y^2}{x^2-y^2}.$

Ex. 3. $\dfrac{3x+2}{a}+\dfrac{4x+3}{b}+\dfrac{5x+4}{c}=\dfrac{3bcx+2bc}{abc}+\dfrac{4acx+3ac}{abc}+\dfrac{5abx+4ab}{abc}$

$$=\frac{bc(3x+2)+ac(4x+3)+ab(5x+4)}{abc}.$$

Ex. 4. $\dfrac{2a}{b}+\dfrac{3a^2}{6}+\dfrac{2b}{a}+\tfrac{1}{2}=\dfrac{24a^2+6a^3b+24b^2+6ab}{12ab}$

$$=\frac{4a^2+a^3b+4b^2+ab}{2ab}.$$

Ex. 5. $\dfrac{x}{x^2-y^2}+\dfrac{y}{x+y}+\dfrac{1}{x-y}=\dfrac{x}{x^2-y^2}+\dfrac{xy-y^2}{x^2-y^2}+\dfrac{x+y}{x^2-y^2}$

$$=\frac{2x+xy-y^2+y}{x^2-y^2}.$$

Ex. 6. Here we multiply the numerator and denominator of the first fraction by its own denominator, to obtain the common denominator.

$$\frac{p}{3my^2-x}+\frac{y-6mpy^2}{(3my^2-x)^2}=\frac{p(3my^2-x)}{(3my^2-x)^2}+\frac{y-6mpy^2}{(3my^2-x)^2}$$

$$=\frac{y-3mpy^2-px}{(3my^2-x)^2}.$$

SUBTRACTION OF FRACTIONS.

Page 32.

Ex. 2. $\frac{7x}{2} - \frac{2x+1}{3} = \frac{21x}{6} - \frac{4x+2}{6} = \frac{17x-2}{6}$.

Ex. 3. $\frac{5x-3}{x+1} - \frac{3x+2}{x-1} = \frac{5x^2-8x+3}{x^2-1} - \frac{3x^2+5x+2}{x^2-1} = \frac{2x^2-13x+1}{x^2-1}$.

Ex. 4. $\frac{1}{x-y} - \frac{1}{x+y} = \frac{x+y}{x^2-y^2} - \frac{x-y}{x^2-y^2} = \frac{2y}{x^2-y^2}$.

Ex. 5. By multiplying the numerator and denominator of the first fraction by $(x+y)$ we have the common denominator $x^2 - y^2$.

$$\therefore \frac{1}{x-y} - \frac{1}{x^2-y^2} = \frac{x+y}{x^2-y^2} - \frac{1}{x^2-y^2} = \frac{x+y-1}{x^2-y^2}.$$

Ex. 6. Here we multiply the two terms of the first fraction by $(x-1)$ for a common denominator.

$$\frac{5x-3}{x+1} - \frac{2x^2-13x+1}{x^2-1} = \frac{5x^2-8x+3}{x^2-1} - \frac{2x^2-13x+1}{x^2-1}$$

$= \frac{3x^2+5x+2}{x^2-1}$, which reduced to its lowest terms is $\frac{3x+2}{x-1}$.

MULTIPLICATION OF FRACTIONS.

Page 33.

Ex. 3. $\frac{a-x^2}{2} \times \frac{2a}{a-x} = \frac{2a^2-2ax^2}{2a-2x} = \frac{a^2-ax^2}{a-x}$.

Ex. 4. $\frac{a+x}{a} \times \frac{a-x}{x} \times \frac{a^2-x^2}{a^2+x^2} = \frac{(a^2-x^2)^2}{ax(a^2+x^2)} = \frac{a^4-2a^2x^2+x^4}{a^3x+ax^3}$.

Ex. 5. $\frac{x^2-y^2}{x} \times \frac{x}{x+y} \times \frac{1}{x-y} = \frac{x(x^2-y^2)}{x(x^2-y^2)} = 1.$

Ex. 6. $\frac{3(a^2-x^2)+a-x}{2} \times \frac{4}{3(a-x)} = \frac{12(a^2-x^2)+4(a-x)}{6(a-x)};$

and dividing the numerator and denominator by $2(a-x)$, we have

$$\frac{12(a^2-x^2)+4(a-x)}{6(a-x)} = \frac{6(a+x)+2}{3}.$$

DIVISION OF FRACTIONS.

Page 34.

Ex. 2. $\frac{ax+b}{a} \times \frac{b}{bx-a} = \frac{abx+b^2}{abx-a^2}.$

Ex. 3. $\frac{6(a+x)+2}{3} \times \frac{3(a-x)}{4} = \frac{18(a^2-x^2)+6(a-x)}{12}$

$$= \frac{3(a^2-x^2)+(a-x)}{2}.$$

Ex. 4. Here we must reduce the dividend to an improper fraction, and shall have

$$a + \frac{2ax-1}{b} = \frac{ab+2ax-1}{b};$$

then,

$$\frac{ab+2ax-1}{b} \times \frac{ax+1}{x-a} = \frac{a^2bx+2a^2x^2+ax+ab-1}{bx-ab}$$

$$= \frac{a^2(bx+2x^2)+a(x+b)-1}{b(x-a)}.$$

Ex. 5. By reducing the divisor to an improper fraction, we find

$$\frac{(a+x)^2}{x} - a = \frac{(a+x)^2 - ax}{x},$$

and 12, or $\dfrac{12}{1} \times \dfrac{x}{(a+x)^2 - ax} = \dfrac{12x}{(a+x)^2 - ax} = \dfrac{12x}{a^2 + ax + x^2}$.

Ex. 6. $\dfrac{a^4 - 2a^2x^2 + x^4}{a^3x + ax^3} \times \dfrac{a^2 + x^2}{a^2 - x^2} = \dfrac{a^6 - a^4x^2 - a^2x^4 + x^6}{a^5x - ax^5} =$

$$\frac{a^2(a^4 - x^4) - x^2(a^4 - x^4)}{ax(a^4 - x^4)} = \frac{(a^2 - x^2)(a^4 - x^4)}{ax(a^4 - x^4)} = \frac{a^2 - x^2}{ax}.$$

INVOLUTION.

Simple Quantities.

Page 36.

Ex. 8. The fifth power of $\dfrac{x^{\frac{1}{2}}}{y^3}$ is $\dfrac{x^{\frac{5}{2}}}{y^{15}}$.

Ex. 9. The seventh power of $-a^{-2}x^{-\frac{1}{2}}$ is $-a^{-14}x^{-\frac{7}{2}}$.

Ex. 10. The fourth power of $-\dfrac{a^m}{2x^n}$ is $\dfrac{a^{4m}}{16x^{4n}}$.

Ex. 11. The nth power of $a^m x^m$ is $a^{mn}x^{mn}$, or $(ax)^{mn}$.†

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† For a very ample discussion of the theory and management of Radical Quantities and Fractional Exponents, see my translation of Lacroix's *Élémens d'Algèbre*, page 217, &c.

Compound Quantities.

Page 37.

Ex. 3.

$$\begin{array}{l}
a - x \\
a - x \\
\hline
a^2 - ax \\
\quad - ax + x^2 \\
\hline
a^2 - 2ax + x^2 \\
a - x \\
\hline
a^3 - 2a^2x + ax^2 \\
\quad - a^2x + 2ax^2 - x^3 \\
\hline
\end{array}$$

$$\therefore (a - x)^3 = a^3 - 3a^2x + 3ax^2 - x^3$$

Ex. 4.

$$\begin{array}{l}
4ax + x + 1 \\
4ax + x + 1 \\
\hline
16a^2x^2 + 4ax^2 + 4ax \\
\qquad\qquad 4ax^2 + x^2 + x \\
\qquad\qquad\qquad 4ax + x + 1 \\
\hline
\end{array}$$

$$\therefore (4ax + x + 1)^2 = 16a^2x^2 + 8ax^2 + 8ax + x^2 + 2x + 1$$

Ex. 5. In Ex. 3. we have found the third power of $(a - x)$

or, $(a - x)^3 = a^3 - 3a^2x + 3ax^2 - x^3$

$$\begin{array}{l}
a - x \\
\hline
a^4 - 3a^3x + 3a^2x^2 - ax^3 \\
\quad - a^3x + 3a^2x^2 - 3ax^3 + x^4 \\
\hline
\end{array}$$

$$\therefore (a - x)^4 = a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$$

Ex. 6. The square of $\sqrt{x^2+y^2}$ is x^2+y^2 (*see note*, p. 5 *of this Key*); and, as the fourth power is equal to the square multiplied by the square, we have

$$\begin{array}{l} x^2+y^2 \\ x^2+y^2 \\ \hline x^4+x^2y^2 \\ \qquad x^2y^2+y^4 \\ \hline \end{array}$$

$$\therefore (\sqrt{x^2+y^2})^4 = x^4+2x^2y^2+y^4$$

Ex. 7.* The square of $\sqrt{a+x}=a+x$; and the sixth power is equal to the square raised to the third power,

$$\begin{array}{l} a+x \\ a+x \\ \hline a^2+ax \\ \qquad ax+x^2 \\ \hline a^2+2ax+x^2 \\ a+x \\ \hline a^3+2a^2x+ax^2 \\ \qquad a^2x+2ax^2+x^3 \\ \hline \end{array}$$

$$\therefore (\sqrt{a+x})^6 = a^3+3a^2x+3ax^2+x^3$$

Ex. 8. Here the fourth power of $\sqrt[4]{x-y}=(\sqrt[4]{x-y})^4=x-y$; and $(\sqrt[4]{x-y})^5=(x-y)\sqrt[4]{x-y}$.

---

* By an error of the press a 3 has been introduced into the radical sign, but the square root is evidently intended, the sixth power of $\sqrt[3]{a+x}$ being equal to $a^2+2ax+x^2$; for, the third power of $\sqrt[3]{a+x}$, that is, $(\sqrt[3]{a+x})^3=a+x$; and the sixth power, being equal to the third power squared, is $(a+x)^2$, or $a^2+2ax+x^2$.

EVOLUTION.

Simple Quantities.

Page 39.

Ex. 6. The square root of $\dfrac{4}{a^{\frac{1}{2}}b^3}$ is $\dfrac{2}{a^{\frac{1}{4}}b^{\frac{3}{2}}}$ or $2a^{-\frac{1}{4}}b^{-\frac{3}{2}}$.

Ex. 7. The fourth root of $a^{-2}b^{-\frac{1}{2}}c$ is $a^{-\frac{1}{2}}b^{-\frac{1}{8}}c^{\frac{1}{4}}$.

Ex. 8. The cube root of $-27a^2b^{\frac{1}{3}}x^{-3}$ is $-3a^{\frac{2}{3}}b^{\frac{1}{9}}x^{-1}$.

Ex. 9. The 5th root of $\dfrac{ab^{10}c^5}{d^2e^{\frac{1}{5}}m^2}$ is $\dfrac{a^{\frac{1}{5}}b^2c}{d^{\frac{2}{5}}e^{\frac{1}{25}}m^{\frac{2}{5}}}$.

Ex. 10. The cube root of $\dfrac{a^{-1}}{b^nx^{\frac{1}{n}}}$ is $\dfrac{a^{-\frac{1}{3}}}{b^{\frac{n}{3}}x^{\frac{1}{3n}}} = \dfrac{1}{a^{\frac{1}{3}}b^{\frac{n}{3}}x^{\frac{1}{3n}}} = a^{-\frac{1}{3}}b^{-\frac{n}{3}}x^{-\frac{1}{3n}}$.

To extract the Square Root of a Compound Quantity.

Page 42.

Ex. 3.

$$
\begin{array}{rl}
 & 4x^4 - 16x^3 + 24x^2 - 16x + 4 \quad (2x^2 - 4x + 2 \\
 & 4x^4 \\
 & \overline{} \\
4x^2 - 4x) & -16x^3 + 24x^2 \\
 & -16x^3 + 16x^2 \\
 & \overline{} \\
4x^2 - 8x + 2) & 8x^2 - 16x + 4 \\
 & 8x^2 - 16x + 4 \\
 & \overline{} \\
 & * \quad * \quad *
\end{array}
$$

Ex. 4. $16x^4 + 24x^3 + 89x^2 + 60x + 100 \quad (4x^2 + 3x + 10$

$$16x^4$$

$$8x^2 + 3x) \quad 24x^3 + 89x^2$$

$$24x^3 + 9x^2$$

$$8x^2 + 6x + 10) \quad 80x^2 + 60x + 100$$

$$80x^2 + 60x + 100$$

$$\cdot \qquad \cdot \qquad \cdot$$

Ex. 5. $1 + x \quad \left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \text{\&c.}\right.$

$$1$$

$$2 + \frac{x}{2}\ ) \ x$$

$$x + \frac{x^2}{4}$$

$$2 + x - \frac{x^2}{8}) - \frac{x^2}{4}$$

$$-\frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{64}$$

$$2 + x - \frac{x^2}{4} + \frac{x^3}{16}) \quad \frac{x^3}{8} - \frac{x^4}{64}$$

$$\frac{x^3}{8} + \frac{\dagger 4x^4}{64} - \frac{x^5}{64} + \frac{x^6}{256}$$

$$2 + x - \frac{x^2}{4} + \frac{x^3}{8} - \frac{5x^4}{128}) - \frac{5x^4}{64} + \frac{\dagger 2x^5}{128} - \frac{\dagger 2x^6}{512}$$

$$-\frac{5x^4}{64} - \frac{5x^5}{128} + \frac{5x^6}{512} - \frac{5x^7}{1024} + \frac{25x^8}{16384}$$

$$2 + x - \frac{x^2}{4} + \frac{x^3}{8} - \frac{5x^4}{64} + \text{\&c.}) \quad \frac{7x^5}{128} - \frac{7x^6}{512} + \frac{5x^7}{1024} - \frac{25x^8}{16384}$$

&c. &c.

---

† The student will observe that these terms have been reduced to the same denominators as the corresponding terms with which they are con-

Ex. 6. $9x^6 - 12x^5 + 10x^4 - 28x^3 + 17x^2 - 8x + 16 \; (3x^3 - 2x^2 + x - 4$

$$
\begin{array}{rl}
 & 9x^6 \\
\hline
6x^3 - 2x^2) & -12x^5 + 10x^4 \\
 & -12x^5 + 4x^4 \\
\hline
6x^3 - 4x^2 + x \;) & 6x^4 - 28x^3 + 17x^2 \\
 & 6x^4 - 4x^3 + x^2 \\
\hline
6x^3 - 4x^2 + 2x - 4) & -24x^3 + 16x^2 - 8x + 16 \\
 & -24x^3 + 16x^2 - 8x + 16 \\
\hline
 & * \quad * \quad * \quad * \\
\hline
\end{array}
$$

To extract the Cube Root of a Compound Quantity.

Page 46.

Ex. 3. $8x^3 + 36x^2 + 54x + 27 \quad (2x + 3$

$$
\begin{array}{rl}
 & 8x^3 \\
\hline
12x^2 + 18x + 9) & 36x^2 + 54x + 27 \\
 & 36x^2 + 54x + 27 \\
\hline
 & * \quad * \quad * \\
\hline
\end{array}
$$

---

nected, in order to effect the operations indicated: viz. the numerator and denominator of the term $\frac{x^4}{16}$ have been multiplied by 4, (making $\frac{4x^4}{64}$) to obtain the same denominator as the term $-\frac{x^4}{64}$; and the difference of the two fractions is then taken; the numerator and denominator of the terms $\frac{x^5}{64}$ and $\frac{x^6}{256}$, have been multiplied by 2, for similar reasons.

Ex. 4. $27x^6 - 54x^5 + 63x^4 - 44x^3 + 21x^2 - 6x + 1\ (3x^2 - 2x + 1$

$$\underline{27x^6}$$

$$27x^4 - 18x^3 + 4x^2)\ -54x^5 + 63x^4 - 44x^3$$

$$\underline{-54x^5 + 36x^4 - 8x^3}$$

$$27x^4 - 36x^3 + 21x^2 - 6x + 1)\ 27x^4 - 36x^3 + 21x^2 - 6x + 1$$

$$\underline{27x^4 - 36x^3 + 21x^2 - 6x + 1}$$

$$*\quad *\quad *\quad *\quad *$$

Ex. 5. $a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3\ (a+b+c$

$$\underline{a^3}$$

$$3a^2 + 3ab + b^2)\ 3a^2b + 3ab^2 + b^3$$

$$\underline{3a^2b + 3ab^2 + b^3}$$

$$3a^2 + 6ab + 3b^2 + 3ac + 3bc + c^2)\ 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$$

$$\underline{3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3}$$

$$*\quad *\quad *\quad *\quad *\quad *$$

SIMPLE EQUATIONS.

Transposition.

Page 48.

Ex. 3. $14 - x = 6x + 22$;

$$\therefore\ -x - 6x = 22 - 14.$$

Ex. 4. $\dfrac{4+x}{3} - x = \dfrac{6(x-2)}{5} - 8$;

$$\therefore\ \frac{4+x}{3} - x - \frac{6(x-2)}{5} = -8.$$

Ex. 5. $$3x + 7 = 23 - 5x + \frac{4x - 1}{2};$$

$$\therefore 3x + 5x - \frac{4x - 1}{2} = 23 - 7.$$

Ex. 6. $$ab + \frac{ax}{b} = a(x - b) + b;$$

$$\therefore \frac{ax}{b} - a(x - b) = b - ab.$$

Ex. 7. $$5x + 8 - \tfrac{1}{2}x = 6 - \tfrac{2}{3}x + ax;$$

$$\therefore 5x - \tfrac{1}{2}x + \tfrac{2}{3}x - ax = 6 - 8.$$

Ex. 8. $$(2 + x)(a - 3) = 13 - 2ax;$$

$$\text{or, } 2a + ax - 3x - 6 = 13 - 2ax;$$

$$\therefore ax + 2ax - 3x = 13 + 6 - 2a.$$

Ex. 9. $$(a + b)(c - x) = (x - a)b;$$

$$\text{or, } ac + bc - ax - bx = bx - ab;$$

$$\therefore -ax - bx - bx = -ab - ac - bc.$$

To clear an Equation of Fractions.

Page 50.

Ex. 3. $$\frac{4(x + 3)}{5} - \frac{21}{4} = \frac{x}{6} - \frac{6x - 8}{7} + 2;$$

$$\therefore 672(x + 3) - 4410 = 140x - 720x + 960 + 1680.$$

Ex. 4. $$\frac{2x + 1}{3} - \frac{1}{7} = \frac{3x + 5}{x - 1};$$

$$\therefore (14x + 7)(x - 1) - 3(x - 1) = 63x + 105.$$

Ex. 5. $$\frac{ax + b}{c} - \frac{a}{b} = \frac{cx + d}{ex};$$

$$\therefore (ax + b)bex - acex = cb(cx + d).$$

Ex. 6. $$\frac{ax}{a+x}+b-\frac{a+x}{x}=0;$$

$$\therefore ax^2+(a+x)\,bx-(a+x)^2=0.$$

Ex. 7. $$\frac{x+3}{4}+6=\frac{2x-1}{3}+\tfrac{1}{2}.$$

Here 12 is the least common multiple of the denominators;
$\therefore$ multiplying by 12, we obtain

$$3x+9+72=8x-4+6.$$

Ex. 8. $$\frac{4a\,(x+1)}{3}+\frac{2a\,(x-2)}{a}=\frac{a+x}{2a}+\frac{2}{3}.$$

Here the least common multiple of the denominators is $6a$;
$\therefore$ multiplying by $6a$,

$$8a^2(x+1)+12a\,(x-2)=3\,(a+x)+4a.$$

Ex. 9. $$a+\frac{3a}{a+x}+2=\frac{4ax}{a-x}+\frac{x}{a^2-x^2}.$$

Here the least common multiple of the denominators is a^2-x^2;
$\therefore$ multiplying by a^2-x^2, we have

$$a\,(a^2-x^2)+3a\,(a-x)+2\,(a^2-x^2)=4ax\,(a+x)+x.$$

Ex. 10. $$\frac{ax}{a-x}+\frac{x}{a+x}=\frac{a}{a-x}+\frac{1}{a^2-x^2}.$$

Here a^2-x^2 is the least common multiple of the denominators;
$\therefore$ multiplying by a^2-x^2, we have

$$ax\,(a+x)+x\,(a-x)=a\,(a+x)+1.$$

Ex. 11. $$\frac{3-x}{2}+\frac{3}{5}=\frac{1}{20}+\frac{x-8}{10}.$$

Here 20 is the least common multiple;
$\therefore$ multiplying by 20, we have

$$30-10x+12=1+2x-16.$$

Ex. 12. $$\frac{a+x}{\sqrt{a^2-x^2}}+\frac{\sqrt{a+x}}{\sqrt{a-x}}=\frac{\sqrt{a-x}}{\sqrt{a+x}}$$

Here the least common multiple is $\sqrt{a^2-x^2}$;
$\therefore$ multiplying by $\sqrt{a^2-x^2}$, we have

$$a+x+a+x=a-x.$$

To clear an Equation of Radical Signs.

Page 52.

Ex. 5. $\sqrt{3-x}+6=8+x.$

by transposing, we have

$$\sqrt{3-x}=8+x-6=2+x;$$

and by squaring, . $3-x=4+4x+x^2.$

Ex. 6. $\sqrt{x}-2=4-3\sqrt{x}.$

by transposing, ... $4\sqrt{x}=6;$

and squaring, ... $16x=36.$

Ex. 7. $24+\sqrt{ax+b}=2x-a.$

by transposing, . . $\sqrt{ax+b}=2x-a-24;$

and by squaring, . $ax+b=4x^2-4ax-96x+a^2+48a+576.$

Ex. 8. $a+\sqrt{-a+\sqrt{x+2}}=3.$

by transposing, . . $\sqrt{-a+\sqrt{x+2}}=3-a;$

or squaring, . . . $-a+\sqrt{x+2}=9-6a+a^2;$

again transposing, . . $\sqrt{x+2}=9-5a+a^2;$

and by squaring, . $x+2=81-90a+43a^2-10a^3+a^4.$

Ex. 9. $\sqrt[3]{a+\sqrt{2ax}}=x.$

by cubing, ... $a+\sqrt{2ax}=x^3;$

then transposing, . . $\sqrt{2ax}=x^3-a;$

and by squaring, . . $2ax=x^6-2ax^3+a^2.$

Ex. 10. $\sqrt{1+\sqrt{x+\sqrt{ax}}}=2.$

by squaring, . . . $1+\sqrt{x+\sqrt{ax}}=4;$

or transposing, . . . $\sqrt{x+\sqrt{ax}}=3;$

then squaring, . . . $x+\sqrt{ax}=9;$

again transposing, . . $\sqrt{ax}=9-x;$

and by squaring, . . $ax=81-18x+x^2.$

Ex. 11. $\sqrt{a-x}+2=6-\sqrt{x}.$

by transposing, . . $\sqrt{a-x}=4-\sqrt{x};$

or squaring, . . $a-x=16-8\sqrt{x}+x;$

then transposing, . $8\sqrt{x}=16+2x-a;$

and by squaring, $64x=256+64x+4x^2-32a-4ax+a^2.$

Ex. 12. $\sqrt[3]{x-4}-1=\sqrt[3]{2+\sqrt{x}}-1.$

by transposing, . . $\sqrt[3]{x-4}=\sqrt[3]{2+\sqrt{x}};$

then cubing, . . . $x-4=2+\sqrt{x};$

again transposing, . . $x-6=\sqrt{x};$

and by squaring, . . $x^2-12x+36=x.$

To solve a Simple Equation containing but one Unknown Quantity.

Page 56.

Ex. 11. Given $\frac{4(x+2)}{3}-1=\frac{3x+1}{2}.$

by clearing, we have $8x+16-6=9x+3;$

and transposing, . . . $8x - 9x = 3 - 16 + 6$;

$\therefore$ (changing signs), ... $x = 7$.

Ex. 12. Given $\dfrac{x-1}{7} + \dfrac{x+4}{3} = x - 3$.

by clearing, . . $3x - 3 + 7x + 28 = 21x - 63$;

then transposing, $3x + 7x - 21x = -63 - 28 + 3$;

and collecting the terms, $-11x = -88$;

$\therefore$ dividing by -11, . . $x = \dfrac{-88}{-11} = 8$.

Ex. 13. Given $\dfrac{x}{2} - \dfrac{x}{3} + 5 = \dfrac{6(x+2)}{8}$.

by clearing, . . . $24x - 16x + 240 = 36x + 72$;

then transposing, . $24x - 16x - 36x = 72 - 240$;

collecting terms, ... $-28x = -168$,

$$\therefore \qquad x = \frac{-168}{-28} = 6.$$

Ex. 14. Given $\dfrac{2}{x+2} + \dfrac{x}{4} = \dfrac{x^2+1}{4x}$.

by clearing, . $32x + 4x^3 + 8x^2 = 4x^3 + 8x^2 + 4x + 8$.

Expunging $4x^3 + 8x^2$ from each side of the equation, we have

$$32x = 4x + 8;$$

and transposing, ... $28x = 8$;

$$\text{whence,} \quad x = \frac{8}{28} = \frac{2}{7}.$$

Ex. 15. Given $\dfrac{1}{a^2 - x^2} - a = \dfrac{ax}{a-x} + \dfrac{a}{a+x}$.

Here $a^2 - x^2$ is the least common multiple of the denominators; multiplying then every term by $a^2 - x^2$, we have

$$1 - a^3 + ax^2 = a^2x + ax^2 + a^2 - ax;$$

by transpo. . $ax - a^2x$, or $x(a - a^2) = a^2 + a^3 - 1$;

$$\therefore x = \frac{a^2 + a^3 - 1}{a - a^2}.$$

Ex. 16. Given $\dfrac{(a-b)x}{2} + \dfrac{x}{3} = \dfrac{ab}{4} + a.$

Here 12 is the least common multiple of the denominators; and, multiplying each term by 12, we obtain

$$6ax - 6bx + 4x = 3ab + 12a;$$

collecting terms, . $x(6a - 6b + 4) = 3a(b + 4)$;

$$\text{whence,} \quad x = \frac{3a(b+4)}{6(a-b)+4}.$$

Ex. 17. Given $\dfrac{2}{3}x^2 + \frac{1}{2}x = x + \dfrac{x^2 + x}{4}.$

Here the least common multiple of the denominators is 12; and, multiplying by 12, gives

$$8x^2 + 6x = 12x + 3x^2 + 3x;$$

dividing by x, . . . $8x + 6 = 12 + 3x + 3$;

then transposing, ... $5x = 9$;

$$\therefore x = \frac{9}{5} = 1\frac{4}{5}.$$

Ex. 18. Given $4abx^2 = \dfrac{3ax^2 - 2bx + ax}{3}.$

Clearing the equation, $12abx^2 = 3ax^2 - 2bx + ax$;

then transposing, . $12abx^2 - 3ax^2 = -2bx + ax$;

dividing by x, . . $x\,(12\,ab - 3a) = a - 2b$;

$$\text{consequently,}\quad x = \frac{a - 2b}{12ab - 3a}.$$

Ex. 19. Given $21 + \frac{3x - 11}{16} = \frac{5\,(x - 1)}{8} + \frac{97 - 7x}{2}$.

Here 16 is the least common multiple of the denominators; therefore, multiplying all the terms by 16, we have

$$336 + 3x - 11 = 10x - 10 + 776 - 56x;$$

by transposing, $3x + 56x - 10x = -10 + 776 - 336 + 11$;

collecting terms, ... $49x = 441$;

$$\text{whence,}\quad x = \frac{441}{49} = 9.$$

Ex. 20. Given $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{5} = 77$.

Here 60 is the least common multiple of the denominators; $\therefore$ multiplying by 60, we find

$$30x + 20x + 15x + 12x = 4620;$$

and collecting the terms, . $77x = 4620$;

$$\text{consequently, we have}\quad x = \frac{4620}{77} = 60.$$

Ex. 21. Given $x + \frac{a}{b}\,x + \frac{c}{b}\,x = m$.

Multiplying every term by b, we shall obtain

$$bx + ax + cx, \quad \text{or} \quad x\,(a + b + c) = bm;$$

$$\text{and therefore,}\quad x = \frac{bm}{a + b + c}.$$

Ex. 22. Given $\sqrt{3x-1}=2$.

Squaring each side, . . . $3x-1=4$;

then by transposing, ... $3x=5$;

and, consequently, $x=\frac{5}{3}$.

Ex. 23. Given $\sqrt{x+x^2}=x+\frac{1}{3}$.

By squaring, . . . $x+x^2=x^2+\frac{2}{3}x+\frac{1}{9}$;

and transposing, . . . $x-\frac{2}{3}x$, or $\frac{1}{3}x=\frac{1}{9}$;

whence we have $x=\frac{3}{9}=\frac{1}{3}$.

Ex. 24. Given $3\sqrt{2x+6}+3=15$.

by transposition, . . $3\sqrt{2x+6}=12$;

or dividing by 3, . . . $\sqrt{2x+6}=4$;

and squaring, ... $2x+6=16$;

again transposing, . . . $2x=10$;

$$\therefore \quad x=\frac{10}{2}=5.$$

Ex. 25. Given $\sqrt[3]{3x+13}-4=0$.

By transposing, we have $\sqrt[3]{3x+13}=4$;

cubing each side, . . . $3x+13=64$;

again transposing, ... $3x=51$;

consequently, $x=\frac{51}{3}=17$.

Ex. 26. Given $\sqrt{x}+3=\sqrt{21+x}$.

By squaring the equation, $x+6\sqrt{x}+9=21+x$;

Subtracting x from each side, and transposing, we find

$$6\sqrt{x}=12;$$

or, dividing by 6, ... $\sqrt{x}=2$;

again squaring, ... $x=4$.

Ex. 27. Given $\dfrac{\sqrt{a^2-y^2}}{\sqrt{a-y}}+y=a+2y$;

or, substituting for the first term its value $\sqrt{a+y}$ (note p. 30, Alg.),

$$\sqrt{a+y}+y=a+2y;$$

by transposing, . . . $\sqrt{a+y}=a+y$;

dividing by $\sqrt{a+y}$, . $1=\sqrt{a+y}$;

then squaring, ... $1=a+y$;

transpo. and changing signs, . $y=1-a$.

Ex. 28. Given $x+\sqrt{a-x}=\dfrac{a}{\sqrt{a-x}}$.

By clearing, . . . $x\sqrt{a-x}+a-x=a$;

then by transposition, . . $x\sqrt{a-x}=x$;

dividing by x, ... $\sqrt{a-x}=1$;

and squaring, ... $a-x=1$;

$\therefore$ (changing signs) . . . $x=a-1$.

Ex. 29. Given $\sqrt{4+\sqrt{x^4-x^2}}=x-2$.

By squaring, . . $4+\sqrt{x^4-x^2}=x^2-4x+4$;

expunging 4 from each side, $\sqrt{x^4-x^2}=x^2-4x$;

again squaring, . . . $x^4-x^2=x^4-8x^3+16x^2$;

then transposing, ... $8x^3=17x^2$;

dividing by x^2, ... $8x = 17$;

consequently, we have $x = \frac{17}{8} = 2\frac{1}{8}$.

Ex. 30. Given $(2+x)^{\frac{1}{2}} + x^{\frac{1}{2}} = 4(2+x)^{-\frac{1}{2}}$;

or, $(2+x)^{\frac{1}{2}} + x^{\frac{1}{2}} = \frac{4}{(2+x)^{\frac{1}{2}}}$ (Art. 38, p. 38.)

By clearing, . . . $2 + x + \{x(2+x)\}^{\frac{1}{2}} = 4$;

by transposition, . . $\{x(2+x)\}^{\frac{1}{2}} = 2 - x$;

then squaring, $x(2+x)$, or $x^2 + 2x = 4 - 4x + x^2$;

and again transposing, . . . $6x = 4$;

whence, consequently, $x = \frac{4}{6} = \frac{2}{3}$.

QUESTIONS PRODUCING SIMPLE EQUATIONS INVOLVING BUT ONE UNKNOWN QUANTITY.

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Ex. 9. Let the first part be . . . x;
then the second will be . $x + 3$,
and the third . . . . . . $x + 3 - 5$,

but, by the question, we have . . . $3x + 6 - 5 = 37$;
or, by transposition, ... $3x = 36$;

consequently, $x = \frac{36}{3} = 12$.

$\therefore$ the first part will be 12 feet, the second 15 feet, and the third 10 feet.

Ex. 10. Let x represent the number, and we shall have

$$\frac{1}{5}x - \frac{1}{6}x = 7;$$

then by clearing, $6x - 5x$, or $x = 210$.

Ex. 11. Let x denote the time they travelled;

then, one will travel $3x$ miles,

while the other travels $7x$ miles;

and, by the question, we have $10x = 150$;

$$\therefore x = \frac{150}{10} = 15.$$

Consequently, one will have travelled 45 miles;

and the other $150 - 45 = 105$ miles.

Ex. 12. Let x represent the less, and $60 - x$ will be the greater. Then, by the question,

$$x(60 - x), \quad \text{or} \quad -x^2 + 60x = 3x^2;$$

or, transposing and dividing by x, $-4x = -60$;

whence we have $x = \dfrac{-60}{-4} = 15$.

$\therefore$ the two parts are 15, and $(60 - 15 =)$ 45.

Ex. 13. Let x be the less, and the greater will be $45 - x$; then, by the question,

$$x(45 - x), \quad \text{or} \quad -x^2 + 45x = 45 - x - x^2;$$

and transposing, ... $46x = 45$;

consequently, $x = \dfrac{45}{46}$.

$\therefore$ the less is $\dfrac{45}{46}$, and the greater $45 - \dfrac{45}{46}$, or $\dfrac{2070}{46} - \dfrac{45}{46} = \dfrac{2025}{46}$.

Ex. 14. Let $2x$ be the first part.

Then, x $\begin{cases} \text{being half of the first,} & 2x = \text{the first part;} \\ \text{.. one third of the second,} & 3x = \text{.. second ..;} \\ \text{.. one fourth of the third,} & 4x = \text{.. third ..;} \end{cases}$

and $9x = 36$.

$$\text{Consequently,} \quad x = \frac{36}{9} = 4;$$

hence, $2x = 8$ is the first part, 12 the second, and 16 the third.

Ex. 15. Let x be the number of volumes in each parcel. Then

the first parcel cost $10x$ sixpences;

the second ...... $18x$ ....;

the third .... $21x$ ....;

and, by the question, $49x$ sixpences $= 490$ sixp. ($=$ £12 : 5*s*.);

$$\therefore x = \frac{490}{49} = 10.$$

Ex. 16. Let x be the number, and we shall have

$$\tfrac{1}{3}x + \tfrac{1}{4}x = \tfrac{1}{6}x + 35;$$

by transposing, . . . $\tfrac{1}{3}x + \tfrac{1}{4}x - \tfrac{1}{6}x = 35$;

and multiplying by 12, the least common multiple of the denominators,

we obtain, . . . $4x + 3x - 2x$, or $5x = 420$;

$$\text{whence,} \quad x = \frac{420}{5} = 84.$$

Ex. 17. Let x be the length of the post; then, by the question, we have

$$\tfrac{1}{3}x + \tfrac{1}{4}x + 10 = x;$$

by clearing, . . . $4x + 3x + 120 = 12x$;

or, transposing, . $(7x - 12x$, or$) - 5x = -120$;

whence, $x = \dfrac{-120}{-5} = 24.$

Ex. 18. Let x be the number of hours required to fill the cistern.

Here we see that the first cock, as it fills the cistern in 8 hours, will supply $\frac{1}{8}$ of the contents of the cistern in one hour; the second will supply $\frac{1}{10}$ in an hour, and the third $\frac{1}{14}$ in an hour;

whence, $\frac{1}{8} + \frac{1}{10} + \frac{1}{14} =$ the quantity supplied in one hour when all three are set running together, and as they continue running for x hours, we multiply the above quantity by x, and shall obtain $\dfrac{x}{8} + \dfrac{x}{10} + \dfrac{x}{14} = 1 =$ the quantity supplied in x hours, that is, the contents of the cistern; and multiplying by 280 the least multiple of the denominators, we have

$$35x + 28x + 20x, \quad \text{or} \quad 83x = 280;$$

$$\text{hence,} \quad x = \frac{280}{83} = 3 \text{ hours, } 22 \text{ min. } 24\frac{48}{83} \text{ sec.}$$

Ex. 19. Let x represent the gentleman's income.

He pays $\frac{2}{3}x$ for board and lodging, and has remaining $x - \frac{2}{3}x$, that is, $\frac{1}{3}x$; he spends $\frac{2}{3}$ of this remainder, that is, $\frac{2}{3}$ of $\frac{1}{3}x = \frac{2}{9}x$ for clothes; and lays by £20;

$$\therefore \quad \frac{2}{3}x + \frac{2}{9}x + 20 = x;$$

and multiplying by 9, the least common multiple of the denominators, we have $$6x + 2x + 180 = 9x;$$

transpo. and changing signs, $x = 180.$

Ex. 20. Let x denote the number of days.

Then $14 + 16 =$ number of miles in one day's journey;

and $14x + 16x$, or $30x = 197 =$ no. of miles in x day's journey;

$$\therefore x = \frac{197}{30} = 6 \text{ days, } 13\tfrac{3}{5} \text{ hours.}$$

Ex. 21. Let the number of beggars be denoted by x.

Then, in one case, he had $3x - 8$; and in the other $2x + 3$.

$\therefore$ $3x - 8 = 2x + 3$; and transposing, $x = 11$.

Ex. 22. Suppose $x =$ the number of shillings he began with. Then $x - \frac{1}{5}x = \frac{4}{5}x$; to which add 4*s*, or $\frac{20}{5}$; and we shall have the amount he then had, viz. $\frac{4x + 20}{5}$; afterwards he lost $\frac{1}{4}$ of this, that is, $\frac{x + 5}{5}$; and therefore had remaining $\frac{3}{4}$, that is, $\frac{3x + 15}{5}$; this latter, with 3 or $\frac{15}{5}$ which he won, will give the amount he had this time, viz. $\frac{3x + 15 + 15}{5}$, or $\frac{3x + 30}{5}$; of this he lost $\frac{1}{6}$, and then had remaining $\frac{5}{6}$, that is, $\frac{15x + 150}{30} = \frac{x + 10}{2}$: and by the question, we have

$$\frac{x + 10}{2} = 20;$$

clearing, ... $x + 10 = 40$;

and transposing, ... $x = 30$.

Or Thus:

Let $5x$ be the number of shillings he began with; then at the first game he lost $\frac{1}{5}$ of his money, that is, x shillings, and afterwards won 4*s*.; consequently, he had left, $4x + 4$. He next lost $\frac{1}{4}$ of this sum, and therefore had remaining $3x + 3$, which, added to the 3*s*. won, makes $3x + 6$; and then losing $\frac{1}{6}$ of this sum, he had, finally, $2\frac{1}{2}x + 5$; but, by the question, this sum is said to be £1, that is,

$$\frac{5}{2}x + 5 = 20;$$

or, clearing, . . . $5x + 10 = 40$;

and transposing, . . . $5x = 30$.

Ex. 23. Let x be the number of gallons required.

then, $(20 \times 9) + (36 \times 11) + 14x = 12(20 + 36 + x)$;

or, $180 + 396 + 14x = 240 + 432 + 12x$;

by transposition,

$$(14x - 12x, \text{ or})\quad 2x = 96 \text{ (or } 240 + 432 - 180 - 396);$$

$$\text{consequently,}\quad x = \frac{96}{2} = 48.$$

Ex. 24. Let x be the number of days.

Then A can do $\frac{1}{a}$ of the work in one day;

and B . . $\frac{1}{b}$ . . . ;

$$\therefore \frac{x}{a} + \frac{x}{b} = 1 = \text{the whole work done in } x \text{ days};$$

by clearing, . . . $bx + ax$, or $x(a + b) = ab$;

$$\text{whence,}\quad x = \frac{ab}{a+b}.$$

Ex. 25. Let x denote the number of days; and here also

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \text{one day's work};$$

$$\text{consequently,}\quad \frac{x}{a} + \frac{x}{b} + \frac{x}{c} + \frac{x}{d} = 1 \text{ (the whole work)};$$

by clearing, . . $bcdx + acdx + abdx + abcx = abcd$;

or $(abc + abd + acd + bcd)x = abcd$;

$$\text{whence,}\quad x = \frac{abcd}{abc + abd + acd + bcd}.$$

Ex. 26. Let x be the required number, and we shall have

$$\sqrt{x+7} = \sqrt{x} + 1;$$

by squaring, ... $x + 7 = x + 2\sqrt{x} + 1$;

transposing, ... $-2\sqrt{x} = -6$;

dividing by 2, ... $-\sqrt{x} = -3$;

again squaring, ... $x = 9$.

Ex. 27. Let x represent the less, the greater will be $x + 6$.

Then by question $\frac{1}{3}x + \frac{x+6}{5} = \frac{x+6}{3} - \frac{1}{5}x$; and multiplying by 15, the least common multiple of the denominators, we have

$$5x + 3x + 18 = 5x + 30 - 3x;$$

by transposition, ($5x + 3x - 5x + 3x$, or) $6x = 12$ (or $30 - 18$);

$$\therefore\ x = \frac{12}{6} = 2.$$

The numbers are, consequently, 2 and 8.

Ex. 28. Let x denote the number of days he was idle. Then, $9x$ pence is the sum spent; and the sum earned is $(24 - x)$ 42 pence $=$ $1008 - 42x$; we have, consequently,

$$1008 - 42x - 9x = 753 \text{ pence } (= \pounds 3.\ 2s.\ 9d.);$$

and by transposition, . . . $-51x = -255$;

$$\text{hence we have } x = \frac{-255}{-51} = 5.$$

Ex. 29. Let x be the number of minute-divisions that the minute hand passes over before overtaking the hour hand, commencing from 5 o'clock; and $\frac{x}{12}$ will be the number of minute-divisions that the hour hand has advanced in the same time; then

$$25 + \frac{x}{12} = x;$$

by clearing, ... $300 + x = 12x$;

and transposing, ... $-11x = -300$;

$$\therefore\ x = \frac{-300}{-11} = 27 \text{ min. } 16\tfrac{4}{11} \text{ sec. past } 5.$$

Ex. 30. Let the first son's share be x pounds.

The second son's share will be $x + \frac{1}{2}x = \frac{3}{2}x$;

then, the amount of these two shares will be $\frac{3}{2}x + x$, or $\frac{5}{2}x$.

The third son's share $= \frac{5}{2}x + \frac{1}{3}\left(\frac{5x}{2}\right)$, or $\frac{5}{2}x + \frac{5}{6}x$, or $\frac{10}{3}x$;

then, the sum of the three shares $= x + \frac{3}{2}x + \frac{10}{3}x$, or $\frac{6x+9x+20x}{6}$,

or $\frac{35x}{6}$.

The fourth son's share $= \frac{35}{6}x + \frac{1}{4}\left(\frac{35}{6}x\right)$, or $\frac{35x}{6} + \frac{35x}{24}$, or

$\frac{175}{24}x$; but the four shares, $x + \frac{3}{2}x + \frac{10}{3}x + \frac{175}{24}x = 315$ pounds; and multiplying by 24 the least common multiple of the denominators, we have

$$24x + 36x + 80x + 175x, \text{ or } 315x = 7560;$$

$$\text{whence } x = \frac{7560}{315} = 24.$$

$\therefore$ The first son's share is $x = 24$; the second son's, $\frac{3}{2}x = 36$;

the third son's, $\frac{10}{3}x = 80$; and the fourth son's, $\frac{175}{24}x = 175$.

To resolve Simple Equations containing two Unknown Quantities.

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Ex. 3. Given $\begin{cases} 6x - 5y = 39, \\ 7x - 3y = 54. \end{cases}$

From the first equation, $x = \frac{39 + 5y}{6}$;

and from the second, $$x = \frac{54 + 3y}{7};$$

$$\therefore \frac{39 + 5y}{6} = \frac{54 + 3y}{7};$$

by clearing, . . . $273 + 35y = 324 + 18y$;

and transposing, $(35y - 18y$, or$)$ $17y = 51$ (or $324 - 273$);

$$\therefore y = \frac{51}{17} = 3;$$

consequently, $x \left(= \frac{39 + 5y}{6}\right) = 9.$

Ex. 4. Given $\begin{cases} \frac{1}{4}x + \frac{1}{2}y = 7, \\ \frac{1}{3}x - \frac{1}{4}y = 2. \end{cases}$

Multiplying the first equation by 4, and the second by 12, we have

$$x + 2y = 28, \quad \text{and} \quad 4x - 3y = 24;$$

from which we obtain $x = 28 - 2y$, and $x = \frac{24 + 3y}{4}$;

$$\therefore 28 - 2y = \frac{24 + 3y}{4};$$

by clearing, . . . $112 - 8y = 24 + 3y$;

and transposing, $(-8y - 3y$, or$)$ $-11y = -88$ (or $24 - 112$);

$$\therefore y = \frac{-88}{-11} = 8;$$

whence $x\ (= 28 - 2y) = 12.$

Ex. 5. Given $\begin{cases} 3x - \frac{1}{4}y = 7, \\ -\frac{1}{2}x + 2y = 14\frac{1}{2}. \end{cases}$

The first multiplied by 4, and the second by 2, will give

$$12x - y = 28, \quad \text{and} \quad -x + 4y = 29;$$

whence $-y = 28 - 12x$, and $y = \frac{29 + x}{4}$; now, by changing signs

in the first of these equations, in order that y may have the plus sign,

we obtain $\quad -28 + 12x = \dfrac{29 + x}{4}$;

by clearing, . . . $-112 + 48x = 29 + x$;

and transposing, $(48x - x$, or$)$ $47x = 141$ (or $29 + 112$);

Consequently, $x = \dfrac{141}{47} = 3$;

and $y\ (= 12x - 28) = 8$.

Second Method.

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Ex. 2. Given $\begin{cases} 7x + 2y = 30, \\ 5x + 3y = 34. \end{cases}$

From the first equation, $y = \dfrac{30 - 7x}{2}$; and substituting this value of y in the second, we have $5x + \dfrac{90 - 21x}{2} = 34$;

by clearing, . . . $10x + 90 - 21x = 68$;

and transposing, $(10x - 21x$, or$)$ $-11x = -22$ (or $68 - 90$);

$\therefore\ x = \dfrac{-22}{-11} = 2$;

and $y \left(= \dfrac{30 - 7x}{2}\right) = 8$.

Ex. 3. Given $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 8, \\ \frac{1}{3}x - \frac{1}{2}y = 1. \end{cases}$

By clearing, we have $3x + 2y = 48$, and $2x - 3y = 6$;

and from the last equation, $\quad x = \dfrac{6 + 3y}{2}$; substituting this value of x

in the preceding, . . $\frac{18+9y}{2}+2y=48$;

by clearing, ... $18+9y+4y=96$;

and transpo. $(9y+4y$, or$)$ $13y=78$ (or $96-18$);

$$\therefore\ y=\frac{78}{13}=6;$$

$$\text{and}\quad x\ (=\frac{6+3y}{2})=12.$$

Ex. 4. Given $\begin{cases}\dfrac{2x+3y}{4}=5,\\[2ex] 2x=\dfrac{54-8y}{3}.\end{cases}$

By clearing, . $2x+3y=20$, and $6x=54-8y$;

from the first of these equations, we have $y=\frac{20-2x}{3}$; and substituting

this value in the second, $6x=54-\frac{160-16x}{3}$;

clearing again, . . . $18x=162-160+16x$;

and transposing, ... $2x=2$;

$$\therefore\ x=1;$$

$$\text{whence}\quad y\ (=\frac{20-2x}{3})=6.$$

Third Method.

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Ex. 2. Given $\begin{cases}6x+5y=128,\\ 3x+4y=\ \ 88.\end{cases}$

By multiplying the second equation by 2, and subtracting the first from it, we have

$$6x + 8y = 176$$
$$6x + 5y = 128$$
$$3y = 48$$
$$\therefore\ y = \frac{48}{3} = 16;$$
$$\text{and}\quad x\left(= \frac{88 - 4y}{3}\right) = 8.$$

Ex. 3. Given $\begin{cases} 7x + 3y = 42, \\ -2x + 8y = 50. \end{cases}$

By multiplying the first equation by 8, and the second by 3, in order that the coefficients of y may be equal, we obtain

$$56x + 24y = 336$$
$$-6x + 24y = 150$$

by subtracting, ... $62x \quad = 186$

$$\therefore\ x = \frac{186}{62} = 3;$$
$$\text{whence}\quad y\left(= \frac{42 - 7x}{3}\right) = 7.$$

ADDITIONAL EXAMPLES.

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Ex. 1. Given $\begin{cases} 5x + 7y = 201, \\ 8x - 3y = 137. \end{cases}$

Multiplying the first equation by 8, and the second by 5, we have

$$40x + 56y = 1608$$
$$40x - 15y = 685$$

by subtracting, . . . $71y = 923$

$$\therefore\ y = \frac{923}{71} = 13;$$

whence $x \left(= \dfrac{137 + 3y}{8}\right) = 22.$

Ex. 2. Given $\begin{cases} -3x + 8y = 29, \\ -4x + 6y = 20. \end{cases}$

Multiplying the first equation by 4, and the second by 3, we have

$$-12x + 32y = 116$$
$$-12x + 18y = 60$$

by subtracting, . . . $\quad 14y = 56$

$$\therefore\ y = \frac{56}{14} = 4;$$

consequently, $x \left(= \dfrac{20 - 6y}{-4}\right) = 1.$

Ex. 3. Given $\begin{cases} 3x - \frac{1}{2}y = 3\frac{1}{2}, \\ -x + 7y = 33. \end{cases}$

Multiplying the first equation by 2, and the second by 6, we have

$$6x - y = 7$$
$$-6x + 42y = 198$$

by adding, ... $\quad 41y = 205;$

$$\therefore y = \frac{205}{41} = 5;$$

and, changing signs, . . $x\ (= 7y - 33) = 2.$

Ex. 4. Given $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 8, \\ \frac{1}{3}x - \frac{1}{5}y = -1. \end{cases}$

Multiplying the first equation by 18, and the second by 30, in order to equalize the coefficients of y, we have

$$9x + 6y = 144$$
$$10x - 6y = -30$$

by adding, ... $19x = 114$;

consequently, $x = \frac{114}{19} = 6$;

whence $y\left(= \frac{-30-10x}{-6}\right) = 15.$

Ex. 5. Given $\begin{cases} \frac{2x}{3} + 5y = 23, \\ 5x + \frac{7y}{4} = -6\frac{1}{4}. \end{cases}$

Multiplying the first equation by 30, and clearing the second,

$$20x + 150y = 690$$
$$20x + 7y = -25$$

by subtraction, . . . $143y = 715$

$$\therefore y = \frac{715}{143} = 5;$$

and $x\left(= \frac{-25-7y}{20}\right) = -3$

Ex. 6. Given $\begin{cases} \frac{x}{2} - 12 = \frac{y}{4} + 8, \\ \frac{x+y}{5} + \frac{x}{3} - 8 = \frac{2y-x}{4} + 27. \end{cases}$

By transposition, in the first equation, we have $\frac{x}{2} - \frac{y}{4} = 20$;

and clearing, ... $2x - y = 80$;

by transposing in the second, $\frac{x+y}{5} + \frac{x}{3} - \frac{2y-x}{4} = 35$;

or, clearing, $12x + 12y + 20x - 30y + 15x = 2100$;

and collecting, . . . $47x - 18y = 2100$;

then, multiplying the equation $2x - y = 80$ by 18, and subtracting it from this latter, we have

$$47x - 18y = 2100$$
$$36x - 18y = 1440$$
$$\overline{11x \qquad = 660;}$$
$$\therefore x = \frac{660}{11} = 60.$$

$$\text{whence } y\ \Big(= \frac{1440 - 36x}{-18}\Big) = 40.$$

Ex. 7. Given $\begin{cases} x + y = a, \\ x^2 - y^2 = b. \end{cases}$

By transposition, the first equation becomes

$x = a - y$; and putting the square of this value of x in the second, $a^2 - 2ay + y^2 - y^2 = b$; then, transposing and changing signs, . . . $2ay = a^2 - b$;

$$\therefore y = \frac{a^2 - b}{2a};$$

whence, $x\ (= a - y) = a - \frac{a^2 - b}{2a} = \frac{2a^2 - a^2 + b}{2a}$ (Art. 25.) $= \frac{a^2 + b}{2a}$.

Or Thus:

$$x + y = a,$$
$$x^2 - y^2, \text{ or } (x + y)(x - y) = b.$$

By substituting in the second equation for $x + y$ its value a, given in the first, we have

$$a\,(x - y) = b;$$

$$\therefore\ x - y = \frac{b}{a};$$

but $$x + y = a;$$

$\therefore$ by adding, ... $2x = a + \dfrac{b}{a} = \dfrac{a^2 + b}{a}$ (Art. 25.)

by subtracting, ... $2y = a - \dfrac{b}{a} = \dfrac{a^2 - b}{a}$;

consequently, $x = \dfrac{a^2 + b}{2a}$, and $y = \dfrac{a^2 - b}{2a}$.

Ex. 8. Given $\begin{cases} b(x + y) = a(x - y), \\ x^2 - y^2 = c. \end{cases}$

By multiplying the first equation by $x + y$, we have

$$b(x + y)^2 = a(x^2 - y^2);$$

and substituting, in this latter, for $x^2 - y^2$ its value given in the second,

$$b(x + y)^2 = ac;$$

$$(x + y)^2 = \frac{ac}{b} = \frac{a^2c}{ab};$$

whence $x + y = \sqrt{\dfrac{a^2c}{ab}} = a\sqrt{\dfrac{c}{ab}}$.

If we multiply the first equation by $x - y$, we shall have

$$b(x^2 - y^2) = a(x - y)^2;$$

by substitution for $x^2 - y^2$, as before, and transposition, we obtain

$$a(x - y)^2 = bc;$$

$$(x - y)^2 = \frac{bc}{a} = \frac{b^2c}{ab};$$

consequently, $x - y = \sqrt{\dfrac{b^2c}{ab}} = b\sqrt{\dfrac{c}{ab}}$.

$\therefore$ by adding to this latter the equation found above, $x + y = a\sqrt{\dfrac{c}{ab}}$.

we have

$$x-y=b\sqrt{\frac{c}{ab}}$$

$$x+y=a\sqrt{\frac{c}{ab}}$$

$$2x \quad =(a+b)\sqrt{\frac{c}{ab}};$$

and by subtracting, . . . $2y=(a-b)\sqrt{\frac{c}{ab}}$;

whence $x=\frac{a+b}{2}\sqrt{\frac{c}{ab}}$, and $y=\frac{a-b}{2}\sqrt{\frac{c}{ab}}$.

QUESTIONS PRODUCING SIMPLE EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES.

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Ex. 5. Let x be the number of half-guineas, and y the number of crowns, then (the half-guinea being 21 sixpences, and the crown 10) we shall have

$$21x+10y=576 \text{ sixpences } (=14l.\ 8s.);$$

also by question . . . . $2y=3x$;

multiplying the second equation by 5, will give $10y=15x$; or, by transposition, $10y-15x=0$; and subtracting this from the first,

we have . . . . . $21x+10y=576$

$$-15x+10y=\ 0$$

$$36x \quad =576$$

$$\therefore x=\frac{576}{36}=16.$$

$$\text{whence} \quad y\left(=\frac{3x}{2}\right)=24.$$

Ex. 6. Let x be the first digit of the two, and it will denote tens; and y the second digit, which will denote units.

Then,
$$10x+y=4x+4y;$$
$$10x+y+18=10y+x;$$

By transposition, we have, from the first equation,

$$6x=3y;$$
$$\therefore\ x=\frac{3}{6}y=\frac{1}{2}y.$$

And from the second equation,

$$9x=9y-18;$$

or, dividing by 9, ... $x=y-2$;

equating these values of x, $y-2=\frac{1}{2}y$,

and transposing, . . $y-\frac{1}{2}y$, or $\frac{1}{2}y=2$;

$$\therefore\ y=4.$$

Consequently, $x\left(=\frac{1}{2}y\right)=2$.

Ex. 7. If we represent the father's age by x, and the son's by y, we shall have

$$\tfrac{1}{4}x+3y=45;$$
$$\tfrac{1}{4}y+3x=111;$$

and by clearing these equations, we obtain

$$x+12y=180,$$
$$12x+\quad y=444;$$

multiplying the second of these equations by 12, we have

$$144x + 12y = 5328,$$

subtracting the first, . $\quad x + 12y = 180,$

$$143x \quad = 5148$$

$$\therefore x = \frac{5148}{143} = 36.$$

And $y\ (= 444 - 12x) = 12.$

Ex. 8. Here, let x denote the numerator, and y the denominator.

Then, $$\frac{2x}{y+7} = \frac{2}{3},$$

$$\frac{x+2}{2y} = \frac{3}{5}.$$

By clearing the first, we have $6x = 2y + 14$,

or transposing, ... $6x - 2y = 14$;

and clearing the second equation, we shall find $5x + 10 = 6y$;

or, by transposition, . . $5x - 6y = -10$; and subtracting this from

first eq. mult. by 3, . . $18x - 6y = 42$

$$13x \quad = 52$$

$$\therefore x = \frac{52}{13} = 4;$$

whence $y\ (= \frac{14 - 6x}{-2}) = 5$. $\therefore \frac{4}{5}$ is the fraction required.

Ex. 9. Let x be the number of days in which the man could drink the whole, and y the number the woman would require.

Then, $\frac{1}{x} + \frac{1}{y} =$ one day's consumption of both.

$\therefore \frac{15}{x} + \frac{15}{y} = 1$ (= the whole quantity, or 15 days' consumption). . 1st Eq.

and, by question, $\frac{6}{x}+\frac{6}{y}+\frac{30}{y}$, or $\frac{6}{x}+\frac{36}{y}=1$ . . . . 2d Eq.

2d Eq. $\times$ 15 . . . $\frac{90}{x}+\frac{540}{y}=15$;

1st Eq. $\times$ 6 . . . . $\frac{90}{x}+\frac{90}{y}=6$;

subtracting, ... $\frac{450}{y}=9$,

clearing, ... $450=9y$.

$$\therefore\ y=\frac{450}{9}=50;$$

and $\frac{15}{x}=1-\frac{15}{y}$, or $\frac{15}{x}=1-\frac{3}{10}$; and, by clearing, $150=10x-3x$;

that is, $7x=150$;

consequently, $x=\frac{150}{7}=21\frac{3}{7}$.

Ex. 10. Let x represent the price in shillings of the wheat, and y the price of the barley. Then,

$$\left.\begin{array}{l} 30x+40y=270 \text{ shillings } (=13l.\ 10s.) \\ 50x+30y=340\ \ .\ .\ \ (=17l.) \end{array}\right\} \text{or dividing by 10} \left\{\begin{array}{l} 3x+4y=27, \\ 5x+3y=34. \end{array}\right.$$

Multiplying the first equation by 3, gives

$$9x+12y=\ 81$$

and the second by 4, . $20x+12y=136$; then, subtracting the preceding from this latter, . $11x\ \ \ =\ 55.$

$$\text{Consequently, } x=\frac{55}{11}=5.$$

$$\text{whence } y\ (=\frac{34-5x}{3})=3.$$

To resolve Simple Equations containing three Unknown Quantities.

First Method.

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Ex. 3. Given $\begin{cases} 7x + 5y + 2z = 79 & \text{. . . . 1st Equation.} \\ 8x + 7y + 9z = 122 & \text{. . . 2d Eq.} \\ x + 4y + 5z = 55 & \text{. . . . 3d Eq.} \end{cases}$

by adding the three, $16x + 16y + 16z = 256$,

and dividing by 16, $x + y + z = 16$ . . . (A).

Then, multiplying equation (A) by 2, and subtracting it from 1st Eq.,

$$7x + 5y + 2z = 79$$
$$2x + 2y + 2z = 32$$

$$5x + 3y \quad = 47 \quad \text{. . . (B).}$$

By multiplying (A) by 5, and subtracting 3d Eq. from it, we have

$$5x + 5y + 5z = 80$$
$$x + 4y + 5z = 55$$

$$4x + y \quad = 25.$$

Multiplying this latter by 3, and subtracting (B) from it, gives

$$12x + 3y = 75$$
$$5x + 3y = 47$$

$$7x \quad = 28$$

consequently, $x = \frac{28}{7} = 4.$

Whence (B), $y\left(=\dfrac{47-5x}{3}\right)=9$, and (A), $z\ (=16-x-y)=3$.

$$\text{Ex. 4. Given}\quad\begin{cases}3x-9y+8z=41 & \text{. . . 1st Eq.}\\ -5x+4y+2z=-20 & \text{. . 2d Eq.}\\ 11x-7y-6z=37 & \text{. . . 3d Eq.}\end{cases}$$

$$\begin{array}{ll}
\text{1st Eq. . . . . . .} & 3x-9y+8z=41\\
\text{2d Eq. mult. by 4,} & -20x+16y+8z=-80\\
\hline
\text{by subtracting, . .} & 23x-25y=121\ \text{. (A)}
\end{array}$$

$$\begin{array}{ll}
\text{3d Eq. . . . .} & 11x-7y-6z=37\\
\text{2d Eq.} \times 3\text{, . .} & -15x+12y+6z=-60\\
\hline
\text{by adding, . . .} & -4x+5y=-23\ \text{. . . (B)}
\end{array}$$

$$\begin{array}{ll}
\text{Equation (A) . . . .} & 23x-25y=121\\
\text{Eq. (B)} \times 5\text{, . . .} & -20x+25y=-115\\
\hline
\text{by adding, ...} & 3x=6
\end{array}$$

consequently, $x=\dfrac{6}{3}=2$.

whence (B), $y\left(=\dfrac{-23+4x}{5}\right)=-3$, and (1st Eq.), $z\left(=\dfrac{41-3x+9y}{8}\right)=1$.

$$\text{Ex. 5. Given}\quad\begin{cases}x+\frac{1}{2}y+\frac{1}{3}z=32,\\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15,\\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12.\end{cases}$$

or, by clearing these equations of fractions, we have

$$\begin{array}{ll}
6x+3y+2z=192 & \text{. . . 1st Equation.}\\
20x+15y+12z=900 & \text{. . . 2d Eq.}\\
30x+24y+20z=1440 & \text{. . . 3d Eq.}
\end{array}$$

Then, 1st Eq. $\times$ 5, $30x + 15y + 10z = 960$

2d Eq. . . $20x + 15y + 12z = 900$;

by subtracting, . . $10x \quad - 2z = 60$ . . . (A).

And 1st Eq. $\times$ 8, . $48x + 24y + 16z = 1536$

3d Eq. . . . $30x + 24y + 20z = 1440$

subtracting, . . . $18x \quad - 4z = 96$ . . . (B).

Eq. (A) $\times$ 2 . . $20x \quad - 4z = 60$

subtracting, . . $2x \quad = 24$

consequently, $x = \frac{24}{2} = 12$.

whence (A), $z\left(= \frac{60-10x}{-2}\right) = 30$, and (1st Eq.), $y\left(= \frac{192-6x-2z}{3}\right) = 20$.

Ex. 6. Given $\begin{cases} \dfrac{x+y}{3} + 2z = 21 \\ \dfrac{y+z}{2} - 3x = -65 \\ \dfrac{3x+y-z}{2} = 38 \end{cases}$

or, by clearing these equations of fractions,

$x + y + 6z = 63$ . . . 1st Eq.

$-6x + y + z = -130$ . . . 2d Eq.

$3x + y - z = 76$ . . . 3d Eq.

1st Eq. . . . . $x + y + 6z = 63$

2d Eq. . . . $-6x + y + z = -130$

subtracting, . . $7x \quad + 5z = 193$ . . . (A).

3d Eq. . . . . . . $3x + y - z = 76$

1st Eq. . . . . . . $x + y + 6z = 63$

subtracting, . . . $2x \quad - 7z = 13$ . . . (B).

Eq. (A) $\times$ 2 . . . $14x + 10z = 386$

Eq. (B) $\times$ 7 . . . $14x - 49z = 91$

subtracting, . . . $59z = 295$

$$\therefore \quad z = \frac{295}{59} = 5;$$

hence (B), $x \left(= \frac{13 + 7z}{2}\right) = 24$, and (1st Eq.), $y \ (= 63 - 6z - x) = 9$.

QUESTIONS PRODUCING SIMPLE EQUATIONS INVOLVING THREE UNKNOWN QUANTITIES.

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Ex. 2. Let x be the first number, y the second, and z the third.

Then, . . . $\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 46$

$\frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 35$

$\frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 28\frac{1}{3}$;

and, clearing these equations of fractions, we have

$6x + 4y + 3z = 552$ . . . 1st Eq.

$20x + 15y + 12z = 2100$ . . . 2d Eq.

$15x + 12y + 10z = 1700$ . . . 3d Eq.

3d Eq. . . . . $15x + 12y + 10z = 1700$

1st Eq. $\times$ 3 . . $18x + 12y + 9z = 1656$

by subtracting . $-3x \quad + z = 44$ . . . (A).

$$\begin{array}{lrl}
\text{2d Eq.} \times 4 \quad . \quad . \quad . & 80x + 60y + 48z = 8400 & \\
\text{1st Eq.} \times 15 \quad . \quad . & 90x + 60y + 45z = 8280 & \\
\hline
\text{subtracting} \quad . \quad . & -10x \qquad + 3z = 120 & . \quad . \quad . \quad (B). \\
\text{Eq. (A)} \times 3 \quad . & -9x \qquad + 3z = 132 & \\
\hline
\text{subtracting,} \quad . \quad . \quad . & x \qquad\qquad = 12 &
\end{array}$$

whence (A), $z\ (= 44 + 3x) = 80$, and (1st Eq.), $y\ (= \dfrac{552 - 6x - 3z}{4}) = 60$.

Ex. 3. Let $z + 14$ be the first share, x the second, y the third, and z the fourth.

$$\tfrac{1}{2}x + \tfrac{1}{2}y + \tfrac{1}{2}z = z + 14$$

$$\frac{z + 14}{3} + \tfrac{1}{3}y + \tfrac{1}{3}z = x$$

$$\frac{z + 14}{4} + \tfrac{1}{4}x + \tfrac{1}{4}z = y$$

By multiplying the first equation by 2, the second by 3, and the third by 4, and then transposing, we shall have

$$\begin{array}{rl}
x + y - z = 28 & . \quad . \quad . \quad \text{1st Eq.} \\
-3x + y + 2z = -14 & . \quad . \quad . \quad \text{2d Eq.} \\
x - 4y + 2z = -14 & . \quad . \quad . \quad \text{3d Eq.}
\end{array}$$

$$\begin{array}{lrl}
\text{Then, 2d Eq.} \quad . & -3x + y + 2z = -14 & \\
\text{3d Eq.} \quad . & x - 4y + 2z = -14 & \\
\hline
\text{by subtracting,} & -4x + 5y \qquad = 0 & . \quad . \quad . \quad (A). \\
\text{2d Eq.} \quad . \quad . \quad . \quad . & -3x + y + 2z = -14 & \\
\text{1st Eq.} \times 2 \quad . & 2x + 2y - 2z = 56 & \\
\hline
\text{by adding} \quad . \quad . & -x + 3y \qquad = 42 & . \quad . \quad . \quad (B).
\end{array}$$

$$\begin{array}{llr} \text{Eq. (B)} \times 4 \quad . \quad . \quad . & -4x + 12y = & 168 \\ \text{Eq. (A)} \quad . \quad . \quad . \quad . & -4x + 5y = & 0 \\ \hline \text{subtracting} \quad . \quad . \quad . \quad . & 7y = & 168 \end{array}$$

$$\therefore\ y = \frac{168}{7} = 24;$$

whence (B with signs changed) $x\,(= -42 + 3y) = 30$, then (1 Eq. with signs changed) $z\,(= -28 + x + y) = 26$, and $z + 14 = 40$.

$\therefore$ the sum of these shares is 120l.

Ex. 4. Let x be the number of crowns, y that of the guineas, and z that of the moidores.

$$\begin{array}{lll} \text{Then,} & 5x + 21y + 27z = 454\ (= 22^{l.}\ 14^{s.}) & .\ .\ \text{1st Eq.} \\ & 21x + 5y + 27z = 726\ (= 36.\ 6) & .\ .\ \text{2d Eq.} \\ & 27x + 21y + 5z = 916\ (= 45.\ 16) & .\ .\ \text{3d Eq.} \end{array}$$

(amounts on the left in $s.$)

$$\begin{array}{lll} \text{2d Eq.} \quad . \quad . \quad . & 21x + 5y + 27z = 726 \\ \text{1st Eq.} \quad . \quad . \quad . & 5x + 21y + 27z = 454 \\ \hline \text{subtracting,} \quad . & 16x - 16y \qquad = 272; & \text{and dividing this equation} \\ \text{by 16,} \quad . \quad . \quad . \quad . & x - y \qquad = 17 & .\ .\ .\ .\ \text{(A).} \end{array}$$

$$\begin{array}{lll} \text{3d Eq.} \times 27 \quad . \quad . & 729x + 567y + 135z = 24732 \\ \text{2d Eq.} \times 5 \quad . \quad . & 105x + 25y + 135z = 3630 \\ \hline \text{subtracting,} \quad . \quad . & 624x + 542y \qquad = 21102; & \text{and dividing this} \\ \text{equation by 2,} \quad . & 312x + 271y \qquad = 10551 & .\ .\ .\ \text{(B).} \\ \text{Eq. (A)} \times 271 \quad . & 271x - 271y \qquad = 4607 \\ \hline \text{adding,} \quad . \quad . \quad . \quad . & 583x \qquad = 15158 \end{array}$$

$$\text{consequently, } x = \frac{15158}{583} = 26.$$

whence (A with signs changed), $y\ (= -17 + x) = 9$, and (1st Eq.),

$$z\ \left(= \frac{454 - 5x - 21y}{27}\right) = 5.$$

ARITHMETICAL PROPORTION AND PROGRESSION.

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Ex. 4. Here, $a = 1$, $d = 2$, $n = 100$, and the formula

$$\tfrac{1}{2}n\ \{2a + (n - 1)\,d\} = \text{S gives}$$

$$50\ (2 + 99 \times 2) = 10000.$$

Ex. 5. Here, $a = 12$, $l = \frac{1}{2}$, $n = 24$; and the formula

$$\frac{n\ (a + l)}{2} = \text{S}, \quad \text{becomes} \quad \frac{24\ (12 + \frac{1}{2})}{2} = 150.$$

Ex. 6. Here, $a = \frac{1}{2}$, $d = \frac{1}{2}$, $n = 25$; and the formula

$\frac{1}{2}n\ \{2a + (n - 1)\,d\} = \text{S}$, becomes $\frac{25}{2}\ \{1 + 24 \times \frac{1}{2}\} = \frac{25}{2} \times 13 = 162\frac{1}{2}$.

Ex. 7. Here, $a = \frac{1}{3}$, $l = \frac{1}{2}$, $n = 5$, whence

$$\tfrac{1}{2} = \tfrac{1}{3} + 4\,d, \text{ that is, } 4d = \tfrac{1}{2} - \tfrac{1}{3}$$

$$\text{consequently, } d = \frac{\frac{1}{2} - \frac{1}{3}}{4} = \frac{\frac{1}{6}}{4} = \frac{1}{24}.$$

and $\frac{1}{3} + \frac{1}{24} = \frac{3}{8}$,

$\frac{3}{8} + \frac{1}{24} = \frac{5}{12}$, $\quad \therefore \frac{3}{8}, \frac{5}{12}$, and $\frac{11}{24}$ are the means required.

$\frac{5}{12} + \frac{1}{24} = \frac{11}{24}$,

Ex. 8. Here, $a = 1$, $n = 23$, $\text{S} = 149\frac{1}{2}$.

$$\text{then } 149\tfrac{1}{2} = \tfrac{23}{2}\ (2 + 22\,d);$$

by clearing . . . . $299 = 23\,(2 + 22d)$;

dividing by 23, ... $13 = 2 + 22d$;

transposing, ... $22d = 13 - 2 = 11$;

$$\therefore\ d = \tfrac{11}{22} = \tfrac{1}{2}.$$

GEOMETRICAL PROPORTION AND PROGRESSION.

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Ex. 4. Here, $a = 9$, $r = 3$, $n = 10$, and the formula

$$\frac{a\,(r^n - 1)}{r - 1} = \mathrm{S},\ \text{becomes}\ \frac{9\,(59049 - 1)}{2} = 265716.$$

Ex. 5. Here, $a = 1$, $r = -\tfrac{1}{2}$, and the formula

$$\frac{a}{1 - r} = \mathrm{S},\ \text{becomes}\ \frac{1}{1 + \tfrac{1}{2}} = \frac{2}{3}.$$

Ex. 6. Here, $a = 1$, $r = \tfrac{1}{3}$, $n = 10$, and

$$\frac{a\,(r^n - 1)}{r - 1} = \mathrm{S}\ \text{gives}\ \frac{1}{59049} - \frac{59049}{59049} = -\frac{59048}{59049};$$

and $\dfrac{1}{3} - \dfrac{3}{3} = -\dfrac{2}{3}$; then $-\dfrac{59048}{59049} \div -\dfrac{2}{3} = \dfrac{177144}{118098} = 1\,\dfrac{9841}{19683}$.

Ex. 7. Here, $a = 1$, $r = -\tfrac{3}{4}$, $n = 6$; and

$$\frac{a\,(r^n - 1)}{r - 1},\ \text{gives}\ \frac{729}{4096} - \frac{4096}{4096} = -\frac{3367}{4096};$$

and $-\dfrac{3}{4} - \dfrac{4}{4} = -\dfrac{7}{4}$; then, $-\dfrac{3367}{4096} \div -\dfrac{7}{4} = \dfrac{13468}{28672} = \dfrac{3367}{7168}$.

Ex. 8. Let the terms be a, ar, ar^2, ar^3, ar^4; and we have

$$a = \tfrac{1}{2},\ ar^4 = \tfrac{9}{8}\quad \therefore \text{by division } \frac{ar^4}{a} = \tfrac{9}{8} \div \tfrac{1}{2} = \tfrac{9}{4};$$

$$\text{that is, } r^4 = \tfrac{9}{4};\ \text{hence } r^2 = \tfrac{3}{2},\ \text{and } r = \sqrt{\tfrac{3}{2}}.$$

$$\therefore \text{the means are } ar = \tfrac{1}{2}\sqrt{\tfrac{3}{2}},\ ar^2 = \tfrac{1}{2} \times \tfrac{3}{2} = \tfrac{3}{4},\ \text{and } ar^3 = \tfrac{3}{4}\sqrt{\tfrac{3}{2}}.$$

Ex. 9. Here, $a = 1$, $r = \dfrac{1}{x}$; and $s = \dfrac{a}{1-r}$ becomes

$$s = \frac{1}{1 - \frac{1}{x}} = \frac{x}{x-1}.$$

Ex. 10. Let the terms be a, ar, ar^2, ar^3, ar^4; we have

$$a = 4,\ ar^4 = 324;\ \text{and by division } \frac{ar^4}{a} = r^4 = 81.$$

$$\text{whence } r = 3;\ \therefore ar = 12,\ ar^2 = 36,\ ar^3 = 108.$$

Ex. 11. Here, $a = 20$, $r = \dfrac{19}{20}$; and $s = \dfrac{a}{1-r}$ becomes

$$\frac{20}{1 - \frac{19}{20}} = \frac{400}{20 - 19} = 400.$$

QUESTIONS IN WHICH PROPORTION IS CONCERNED.

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Ex. 4. Let x denote the number of bushels of wheat, and y that of the barley.

Then,	$10x + 4y$	is the cost price,
	$11x + 11y$	. . selling price;
subtracting, . .	$x + 7y$	. . profit.

hence, we have . $10x + 4y : x + 7y :: 100 : 43\frac{3}{4}$

by clearing, . . $40x + 16y : 4x + 28y :: 400 : 175,$

$\times$ Extremes and means, $7000x + 2800y = 1600x + 11200y;$

$\div$ by 200 . . . . $35x + 14y = 8x + 56y;$

transpo. and $\div$ by 3, . . . $9x = 14y.$

$$x : y :: 14 : 9.$$

Ex. 5. Let x be the first digit, and y the second.

Then we have $x + y : 10x + y :: 4 : 13$

and $y - x : 10y + x :: 2 : 31$

$\times$ Extr. and means (Eq. 1), . $13x + 13y = 40x + 4y;$

transpo. and $\div$ by 9, . . . $y = 3x;$

which will also be found from the second equation,

$$\therefore\ x : y :: 1 : 3 :: 2 : 6 :: 3 : 9;$$

hence, there are three numbers which fulfil the conditions of the question, viz. 13, 26, and 39.

Ex. 6. Here, the hour-hand passes over 5 divisions or minutes, while the minute-hand passes over 60; and putting x for the number of divisions which the hour-hand passes over before they arrive at the required position, we have

$$x : 5 :: 25 + x : 60,$$

$\times$ Extremes and means, $60x = 125 + 5x;$

transposing . . . . . . $55x = 125;$

$$\therefore\ x = \frac{125}{55} = 2\frac{3}{11}.$$

and $25 + 2\frac{3}{11} = 27$ min. $16\frac{4}{11}$ sec.

Ex. 7. Let x be the first number, and y the other.

Then $x+y : x-y :: 3 : 2$ . . . (1).

$x-y \,.\; xy :: 2 : 5$ . . . (2).

$\times$ Extremes and means (1), $2x+2y=3x-3y$;

transposing, ... $5y=x$.

and, substituting this value of y in (2) we have

$$5y-y, \text{ or } 4y : 5y^2 :: 2 : 5.$$

$\times$ Extremes and means, $10y^2=20y$;

$\div$ by y, ... $10y=20$;

$$y=\frac{20}{10}=2.$$

whence, $x\,(=5y)=10$.

Ex. 8. Let $\frac{1}{2}x$ be one number, and $\frac{2}{3}x$ the other,

then, $\frac{1}{2}x+6 : \frac{2}{3}x+5 :: \frac{2}{5} : \frac{1}{2}$;

$\times$ Extremes and means, $\frac{4}{15}x+2=\frac{1}{4}x+3$;

transposing, ... $\frac{4}{15}x-\frac{1}{4}x$, or $\frac{1}{60}x=1$;

$$\therefore x=60.$$

Consequently, $\frac{1}{2}x=30$, and $\frac{2}{3}x=40$.

Ex. 9. Let x be the number of gallons of the best brandy, and y that of the inferior; then reducing the prices to sixpences, we have, by the question,

$$81x+72y=79x+79y;$$

transposing, ... $2x=7y$.

$$\therefore x : y :: 7 : 2.$$

Ex. 10. Let x be one number, and y the other.

$$x + y : x - y :: s : d \quad . \quad . \quad . \quad (1).$$

$$x - y \,.\quad xy \;:: d : p \quad . \quad . \quad . \quad (2).$$

$\times$ Extremes and means (1), $sx - sy = dx + dy$;

transposing . . . . $(s - d)\, x = (s + d)\, y$;

$$\therefore x = \frac{(s + d)\, y}{s - d}.$$

Substituting in (2) this value of x, we shall have

$$\frac{(s + d)\, y}{s - d} - y : \frac{(s + d)\, y^2}{s - d} :: d : p,$$

$\times$ Extremes and means, $\frac{s + d}{s - d}\, dy^2 = \left(\frac{s + d}{s - d} - 1\right) py$;

$\times$ by $s - d$, and $\div$ by y, $(s + d)\, dy = (s + d - s + d)\, p$;

that is, $(s + d)\, dy = 2dp$

$\div$ by d, ... $(s + d)\, y = 2p$;

$$\therefore y = \frac{2p}{s + d}.$$

$$\text{whence,}\quad x \left(= \frac{(s + d)\, y}{s - d},\ \text{or}\ \frac{(s + d)\, \frac{2p}{s + d}}{s - d}\right) = \frac{2p}{s - d}.$$

Ex. 11. Let the greyhound take x leaps; then the hare (in the same time) takes $\frac{4}{3}x$; so that the hare will have taken altogether $\frac{4}{3}x + 50$ leaps. Now, by the question, the greyhound's leaps are to the hare's as 2 to 3; therefore,

$$x : \tfrac{4}{3}x + 50 :: 2 : 3$$

$\times$ Extremes and means, $3x = \frac{8}{3}x + 100$;

clearing, ... $9x = 8x + 300$;

and, transposing, ... $x = 300$.

Ex. 12. Let e, f, g, represent days or hours.

$$\text{A can do } \frac{a}{e} \text{ in one day or hour.}$$

$$\text{B} \ \ldots\ldots \ \frac{b}{f} \ \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\text{C} \ \ldots\ldots \ \frac{c}{g} \ \ldots\ldots\ldots\ldots\ldots\ldots$$

$\therefore$ together they produce the effect $\frac{a}{e} + \frac{b}{f} + \frac{c}{g}$ in one day or hour.

Consequently, $\frac{a}{e} + \frac{b}{f} + \frac{c}{g} : 1 :: d : \dfrac{d}{\frac{a}{e} + \frac{b}{f} + \frac{c}{g}}$; for, since the product of the two means is equal to that of the two extremes, the fourth term is found by dividing the product of the means by the first term.

Ex. 13. Let the progression be denoted by x, xy, xy^2, xy^3.

$$\text{Then by question,} \quad . \quad x + xy^2 = 148$$

$$\text{and} \quad . \quad . \quad . \quad xy + xy^3 = 888$$

$$\div \text{ 2d eq. by 1st,} \quad . \quad \frac{xy\,(1+y^2)}{x\;(1+y^2)}, \text{ or } \frac{xy}{x} = \frac{888}{148} = 6;$$

$$\therefore y = 6;$$

and, substituting this value of y, in the first equation, we have

$$x + 36x, \text{ or } 37x = 148;$$

$$\text{consequently, } x = \frac{148}{37} = 4.$$

hence the numbers are 4, 24, 144, and 864, which give $4 + 144 = 148$.

$$\text{and } 24 + 864 = 888.$$

Ex. 14. Let x be A's stock, and y B's stock.

$$\text{Then,}\quad x+150 : y-50 :: 3 : 2 \quad \ldots \quad (1).$$
$$x-50 : y+100 :: 5 : 9 \quad \ldots \quad (2).$$

$\times$ Extremes and means (1), $2x+300 = 3y-150$

$\times$ .... (2), $9x-450 = 5y+500$

$\times$ (2) by 3, ... $27x-1350 = 15y+1500$

$\times$ (1) by 5, ... $10x+1500 = 15y-750$

Subtracting, ... $17x-2850 = 2250$

transposing, ... $17x = 5100$;

$$\therefore\ x = \frac{5100}{17} = 300.$$

$$\text{whence}\quad y\ (= \frac{2x+300+150}{3}) = 350.$$

QUADRATIC EQUATIONS.

QUADRATICS INVOLVING ONLY ONE UNKNOWN QUANTITY.

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Ex. 9. Given $8x^2+6 = 7x+171$.

by transposing, . . . $8x^2-7x = 165$;

$\div$ by 8, ... $x^2-\frac{7}{8}x = \frac{165}{8}$;

completing the square,

$$x^2-\frac{7}{8}x+\left(\frac{7}{16}\right)^2 = \frac{165}{8}+\left(\frac{7}{16}\right)^2 = \frac{5280}{256}+\frac{49}{256} = \frac{5329}{256};$$

Extracting the square root, $x - \frac{7}{16} = \pm\sqrt{\frac{5329}{256}} = \pm\frac{73}{16}$;

transposing, ... $x = \frac{7}{16} \pm \frac{73}{16}$;

that is, $x = \frac{80}{16} = 5$, or $\frac{-66}{16} = -4\frac{1}{8}$.

Ex. 10. Given $3x^2 = 42 - 5x$.

transposing, ... $3x^2 + 5x = 42$;

$\div$ by 3, ... $x^2 + \frac{5}{3}x = 14$;

completing, $x^2 + \frac{5}{3}x + \left(\frac{5}{6}\right)^2 = 14 + \left(\frac{5}{6}\right)^2 = \frac{529}{36}$;

Extracting the root, $x + \frac{5}{6} = \pm\sqrt{\frac{529}{36}} = \pm\frac{23}{6}$;

transposing, ... $x = -\frac{5}{6} \pm \frac{23}{6}$.

that is, $x = 3$, or $-4\frac{2}{3}$.

Ex. 11. Given $4x - \frac{36 - x}{x} = 46$.

$\times$ by x, ... $4x^2 - 36 + x = 46x$;

transposing, ... $4x^2 - 45x = 36$;

$\div$ by 4, ... $x^2 - \frac{45}{4}x = 9$;

completing, $x^2 - \frac{45}{4}x + \left(\frac{45}{8}\right)^2 = 9 + \left(\frac{45}{8}\right)^2 = \frac{2601}{64}$;

extracting the root, $x - \frac{45}{8} = \pm\sqrt{\frac{2601}{64}} = \pm\frac{51}{8}$;

transposing, ... $x = \frac{45}{8} \pm \frac{51}{8}$.

that is, $x = 12$, or $-\frac{3}{4}$.

Ex. 12. Given $\dfrac{6(2x-11)}{x-3}+x-2=24-3x.$

clearing, $12x-66+x^2-3x-2x+6=24x-72-3x^2+9x$

transposing, ... $4x^2-26x=-12;$

$\div$ by 4, ... $x^2-\dfrac{26}{4}x=-3;$

completing, $x^2-\dfrac{26}{4}x+\left(\dfrac{13}{4}\right)^2=-3+\left(\dfrac{13}{4}\right)^2=\dfrac{121}{16};$

extracting the root, $x-\dfrac{13}{4}=\pm\sqrt{\dfrac{121}{16}}=\pm\dfrac{11}{4};$

transposing, . . $x=\dfrac{13}{4}\pm\dfrac{11}{4}=6$, or $\frac{1}{2}$.

Ex. 13. Given $\dfrac{120}{3x+1}+\dfrac{90}{x}=42.$

$\div$ by 6, ... $\dfrac{20}{3x+1}+\dfrac{15}{x}=7;$

clearing of fractions, $20x+45x+15=21x^2+7x;$

transposing and changing signs, $21x^2-58x=15;$

$\div$ by 21, ... $x^2-\dfrac{58}{21}x=\dfrac{15}{21};$

completing, $x^2-\dfrac{58}{21}x+\left(\dfrac{29}{21}\right)^2=\dfrac{15}{21}+\left(\dfrac{29}{21}\right)^2=\dfrac{1156}{441};$

extracting the root, $x-\dfrac{29}{21}=\pm\sqrt{\dfrac{1156}{441}}=\pm\dfrac{34}{21};$

transposing, . . $x=\pm\dfrac{34}{21}+\dfrac{29}{21}=3$, or $-\dfrac{5}{21}$.

Ex. 14. Given $x^3+(19-x)^3=1843.$

that is, . . $x^3+6859-1083x+57x^2-x^3=184$

transposing, . . $57x^2 - 1083x = -5016$;

$\div$ by 57, ... $x^2 - 19x = -88$;

completing, $x^2 - 19x + \left(\frac{19}{2}\right)^2 = -88 + \left(\frac{19}{2}\right)^2 = \frac{9}{4}$;

extracting the root, $x - \frac{19}{2} = \pm\sqrt{\frac{9}{4}} = \pm\frac{3}{2}$;

transposing, . . $x = \pm\frac{3}{2} + \frac{19}{2} = 11$, or 8.

Ex. 15. Given $325 + x : x :: 245 + x : 60$.

$\times$ Extremes and means, $19500 + 60x = 245x + x^2$;

transpo. and changing signs, $x^2 + 185x = 19500$;

completing, $x^2 + 185x + \left(\frac{185}{2}\right)^2 = 19500 + \left(\frac{185}{2}\right)^2 = \frac{112225}{4}$;

extracting the root, $x + \frac{185}{2} = \pm\sqrt{\frac{112225}{4}} = \pm\frac{335}{2}$;

transposing, . . . $x = -\frac{185}{2} \pm \frac{335}{2} = 75$, or -260.

Ex. 16. This question (which expresses only an identity), is to be omitted. (See errata.)

Ex. 17. Given $\{34 - (x - 1)3\}\frac{x}{2} = 57$;

$$\text{or } (34 - 3x + 3)\frac{x}{2} = 57:$$

that is, . . $\frac{34x}{2} - \frac{3x^2}{2} + \frac{3x}{2}$, or $-\frac{3x^2}{2} + \frac{37x}{2} = 57$;

$\times$ by $\frac{2}{3}$, and changing signs, $x^2 - \frac{37}{3}x = -38$;

completing, $x^2 - \frac{37}{3}x + \left(\frac{37}{6}\right)^2 = -38 + \left(\frac{37}{6}\right)^2 = \frac{1}{36}$;

extracting the root, $x - \frac{37}{6} = \pm\sqrt{\frac{1}{36}} = \pm\frac{1}{6}$;

transposing, . . $x = \frac{37 \pm 1}{6} = 6\frac{1}{3}$, or 6.

Ex. 18. Given $\frac{10}{x} - \frac{14 - 2x}{x^2} = \frac{22}{9}$.

$\div$ by 2, ... $\frac{5}{x} - \frac{7 - x}{x^2} = \frac{11}{9}$;

clearing and $\div$ by x, $45x - 63 + 9x = 11x^2$;

transpo. and changing signs, $11x^2 - 54x = -63$;

$\div$ by 11, ... $x^2 - \frac{54}{11}x = -\frac{63}{11}$;

completing, $x^2 - \frac{54}{11}x + \left(\frac{27}{11}\right)^2 = -\frac{63}{11} + \left(\frac{27}{11}\right)^2 = \frac{36}{121}$;

extracting the root, $x - \frac{27}{11} = \pm\sqrt{\frac{36}{121}} = \pm\frac{6}{11}$;

transposing, . . . $x = \frac{27 \pm 6}{11} = 3$, or $\frac{21}{11}$.

Ex. 19. Given $\frac{y^3 - 10y^2 + 1}{y^2 - 6y + 9} = y - 3$.

Clearing, $y^3 - 10y^2 + 1 = y^3 - 6y^2 + 9y - 3y^2 + 18y - 27$;

transposing, . . . $-y^2 - 27y = -28$;

compl. and ch. signs, $y^2 + 27y + \left(\frac{27}{2}\right)^2 = 28 + \left(\frac{27}{2}\right)^2 = \frac{841}{4}$;

extracting the root, $y + \frac{27}{2} = \pm\sqrt{\frac{841}{4}} = \pm\frac{29}{2}$;

transposing, . . $y=\frac{-27\pm 29}{2}=1$, or -28.

Ex. 20. Given $\frac{6x^2-23x+10}{9-2x}=-7x+42$.

Clearing, $6x^2-23x+10=-63x+378+14x^2-84x$;

transposing, . . . $-8x^2+124x=368$;

$\div$ by 8, . . . $-x^2+\frac{124}{8}x$, or $-x^2+\frac{31}{2}x=46$;

compl. and ch. signs, $x^2-\frac{31}{2}x+\left(\frac{31}{4}\right)^2=-46+\left(\frac{31}{4}\right)^2=\frac{225}{16}$;

extracting the root, $x-\frac{31}{4}=\pm\sqrt{\frac{225}{16}}=\pm\frac{15}{4}$;

transposing, . . $x=\frac{31\pm 15}{4}=11\frac{1}{2}$, or 4.

Ex. 21. Given $x+\frac{\sqrt{x-3}}{2}=8$.

Transposing, ... $\frac{\sqrt{x-3}}{2}=8-x$;

squaring, ... $\frac{x-3}{4}=64-16x+x^2$;

clearing, ... $x-3=256-64x+4x^2$;

transpo. and ch. signs, $4x^2-65x=-259$;

$\div$ by 4 and compl., $x^2-\frac{65}{4}x+\left(\frac{65}{8}\right)^2=-\frac{259}{4}+\left(\frac{65}{8}\right)^2=\frac{81}{64}$;

extracting the root, $x-\frac{65}{8}=\pm\sqrt{\frac{81}{64}}=\pm\frac{9}{8}$;

transposing, . . . $x=\frac{65\pm 9}{8}=9\frac{1}{4}$, or 7.

Ex. 22. Given $2x + \frac{x^3}{\sqrt{2x^4 - 3x^3}} = 2x(x + 1)$, or $2x^2 + 2x$.

Cancelling $2x$ and $\div$ by x, . $\frac{x^2}{\sqrt{2x^4 - 3x^3}} = 2x$;

$(x = \sqrt{x^2})$, ... $\frac{x^2}{x\sqrt{2x^2 - 3x}} = 2x$;

$\div$ numer. and deno. by x, $\frac{x}{\sqrt{2x^2 - 3x}} = 2x$;

$\div$ by x, ... $\frac{1}{\sqrt{2x^2 - 3x}} = 2$;

squaring, ... $\frac{1}{2x^2 - 3x} = 4$;

clearing and chang. sides, $8x^2 - 12x = 1$;

$\div$ by 8, . . . $x^2 - \frac{12}{8}x$, or $x^2 - \frac{3}{2}x = \frac{1}{8}$;

completing, . . $x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 = \frac{1}{8} + \left(\frac{3}{4}\right)^2 = \frac{11}{16}$;

extracting the root, $x - \frac{3}{4} = \pm\sqrt{\frac{11}{16}} = \pm\frac{\sqrt{11}}{4}$;

transposing, . $x = \frac{3 \pm \sqrt{11}}{4} = \frac{3 + \sqrt{11}}{4}$, or $\frac{3 - \sqrt{11}}{4}$.

Ex. 23. Given $\sqrt{4 + \sqrt{2x^3 + x^2}} = \frac{x + 4}{2}$.

Squaring, $4 + \sqrt{2x^3 + x^2} = \frac{x^2 + 8x + 16}{4} = \frac{x^2 + 8x}{4} + 4$;

cancelling 4 and clearing, . $4\sqrt{2x^3 + x^2} = x^2 + 8x$;

$(x = \sqrt{x^2})$, . . . $4x\sqrt{2x + 1} = x^2 + 8x$;

$\div$ by x, ... $4\sqrt{2x+1} = x + 8$;

squaring, . . . $32x + 16 = x^2 + 16x + 64$;

transpo. and changing signs, $x^2 - 16x = -48$;

completing, . . $x^2 - 16x + 64 = -48 + 64 = 16$;

extracting the root, $x - 8 = \pm\sqrt{16} = \pm 4$;

transposing, . . . $x = \pm 4 + 8 = 12$, or 4.

Ex. 24. Given* $x^{\frac{3}{2}} + x^{\frac{5}{2}} = 6x^{\frac{1}{2}}$.

$\div$ by $x^{\frac{1}{2}}$, ... $x + x^2 = 6$;

completing, . . $x^2 + x + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$;

extracting the root, $x + \frac{1}{2} = \pm\sqrt{\frac{25}{4}} = \pm\frac{5}{2}$;

transposing, . . . $x = \frac{-1 \pm 5}{2} = 2$, or -3.

Ex. 25. Given $\sqrt[3]{x^3 - a^3} = x - b$.

By cubing, . . $x^3 - a^3 = x^3 - 3bx^2 + 3b^2x - b^3$;

cancelling x^3, and transpo., $3bx^2 - 3b^2x = a^3 - b^3$;

$\div$ by $3b$, ... $x^2 - bx = \frac{a^3 - b^3}{3b}$;

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* The learner must conceive the above quantities, $x^{\frac{3}{2}} + x^{\frac{5}{2}} = 6x^{\frac{1}{2}}$, to be expressed thus: $x^{\frac{2}{2}+\frac{1}{2}} + x^{\frac{4}{2}+\frac{1}{2}} = 6x^{\frac{1}{2}}$; and then, by subtracting $\frac{1}{2}$ from each of the indices (which is in effect dividing by the square root of x), he will find $x^{\frac{2}{2}} + x^{\frac{4}{2}} = 6$, that is, $x + x^2 = 6$, as above; for, $x^{\frac{2}{2}} = x^1$, or x, and $x^{\frac{4}{2}} = x^2$.

completing,

$$x^2 - bx + \frac{b^2}{4} = \frac{a^3 - b^3}{3b} + \frac{b^2}{4} = \frac{4a^3 - 4b^3 + 3b^3}{12b} = \frac{4a^3 - b^3}{12b};$$

extracting the root, $x - \frac{b}{2} = \pm\sqrt{\frac{4a^3 - b^3}{12b}}$;

transposing, ... $x = \frac{b}{2} \pm \sqrt{\frac{4a^3 - b^3}{12b}}$.

Ex. 26. Given $\frac{x + a}{x} + \frac{x}{x + a} = b$.

clearing and chan. sides, $bx^2 + bax = x^2 + 2ax + a^2 + x^2$;

transposing, . . $(b - 2)x^2 + (b - 2)ax = a^2$;

$\div$ by $(b - 2)$, . . . $x^2 + ax = \frac{a^2}{b - 2}$;

completing,

$$x^2 + ax + \frac{a^2}{4} = \frac{a^2}{b - 2} + \frac{a^2}{4} = \frac{4a^2 + (b - 2)a^2}{4(b - 2)} = \frac{a^2}{4}\left(\frac{b + 2}{b - 2}\right);$$

extracting the root, $x + \frac{a}{2} = \pm\sqrt{\frac{a^2}{4}\left(\frac{b + 2}{b - 2}\right)} = \pm\frac{a}{2}\sqrt{\frac{b + 2}{b - 2}}$;

$$\therefore x = -\frac{a}{2} \pm \frac{a}{2}\sqrt{\frac{b + 2}{b - 2}} = \frac{a}{2}\left\{-1 \pm \sqrt{\frac{b + 2}{b - 2}}\right\}.$$

Ex. 27. Given $\sqrt{a + x} + \sqrt{b + x} = \sqrt{a + b + 2x}$.

Squaring, $a + x + 2\sqrt{(a + x)(b + x)} + b + x = a + b + 2x$;

cancelling $a + b + 2x$,..$2\sqrt{(a + x)(b + x)} = 0$;

$\div$ by 2, and squaring, $x^2 + ax + bx + ab = 0$;

or, transposing, . . $x^2 + (a + b)x = -ab$;

completing,

$$x^2 + (a + b)x + \frac{(a + b)^2}{4} = -ab + \frac{(a + b)^2}{4} = \frac{a^2 - 2ab + b^2}{4};$$

extracting the root, $x+\dfrac{a+b}{2}=\pm\sqrt{\dfrac{a^2-2ab+b^2}{4}}=\pm\dfrac{a-b}{2}$;

and transposing, $x=\dfrac{-(a+b)\pm(a-b)}{2}=-a$, or $-b$.

Ex. 28. Given $\sqrt{x-\dfrac{1}{x}}+\sqrt{1-\dfrac{1}{x}}=x$.

$\times$ by $x^{\frac{1}{2}}$, . . . $\sqrt{x^2-1}+\sqrt{x-1}=x^{\frac{3}{2}}$; . . . $\left\{\begin{array}{l} x^{\frac{1}{2}}=\sqrt{x} \\ x = \quad x^{\frac{2}{2}} \\ x^{\frac{2}{2}+\frac{1}{2}}=x^{\frac{3}{2}} \end{array}\right\}$

transposing, . . $\sqrt{x^2-1}=x^{\frac{3}{2}}-\sqrt{x-1}$;

squaring, $x^2-1=x^3-2x^{\frac{3}{2}}\sqrt{x-1}+x-1$; . . . $(x^{\frac{3}{2}}\times x^{\frac{3}{2}}=x^{\frac{6}{2}}=x^3)$

cancelling -1 and $\div$ by x, $x=x^2-2x^{\frac{1}{2}}\sqrt{x-1}+1$;

transpo. and ch. signs, $x^2-x-2\sqrt{x^2-x}+1=0$; . . $(x^{\frac{1}{2}}=\sqrt{x})$

extracting the root, . . $\sqrt{x^2-x}-1=0$;

transposing, ... $\sqrt{x^2-x}=1$;

squaring, ... $x^2-x=1$;

completing, . . . $x^2-x+\dfrac{1}{4}=1+\dfrac{1}{4}=\dfrac{5}{4}$;

extracting the root, . . $x-\dfrac{1}{2}=\pm\sqrt{\dfrac{5}{4}}=\pm\dfrac{\sqrt{5}}{2}$;

transposing, ... $x=\dfrac{1\pm\sqrt{5}}{2}$.

Ex. 29. Given $x-\dfrac{12+8\sqrt{x}}{x-5}=0$.

Clear. and transpo., . $x^2-5x=12+8\sqrt{x}$;

adding x, . . . $x^2 - 4x = 12 + 8\sqrt{x} + x$;

completing, . . $x^2 - 4x + 4 = 16 + 8\sqrt{x} + x$;

extracting the root, $x - 2 = \sqrt{16 + 8\sqrt{x} + x} = 4 + \sqrt{x}$;

transposing, ... $x - \sqrt{x} = 6$;

completing, . . $x - \sqrt{x} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$;

extracting the root, $\sqrt{x} - \frac{1}{2} = \pm\sqrt{\frac{25}{4}} = \pm\frac{5}{2}$;

transposing, . . . $\sqrt{x} = \frac{1 \pm 5}{2} = 3$, or -2;

squaring, ... $x = 9$, or 4.

Ex. 30. Given $x + \sqrt{x} : x - \sqrt{x} :: 3\sqrt{x} + 6 : 2\sqrt{x}$.

$\times$ Extremes and means, $2\sqrt{x}\,(x + \sqrt{x}) = (3\sqrt{x} + 6)(x - \sqrt{x})$;

$\div$ by $\sqrt{x}$, . $2(x + \sqrt{x}) = (3\sqrt{x} + 6)(\sqrt{x} - 1)$;

that is, $2x + 2\sqrt{x} = 3x + 6\sqrt{x} - 3\sqrt{x} - 6$;

transpo. and changing signs, $x + \sqrt{x} = 6$;

completing, ... $x + \sqrt{x} + \frac{1}{4} = \frac{25}{4}$;

extracting the root, . . $\sqrt{x} + \frac{1}{2} = \pm\frac{5}{2}$;

transposing, . . . $\sqrt{x} = \frac{-1 \pm 5}{2} = 2$, or -3;

squaring, ... $x = 4$, or 9.

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Ex. 5. Given $(2x+6)^{\frac{1}{2}} + (2x+6)^{\frac{1}{4}} = 6$.

by putting $(2x+6)^{\frac{1}{2}} = y^2$, the equation becomes

$$y^2 + y = 6;$$

completing, ... $y^2 + y + \frac{1}{4} = \frac{25}{4}$;

extracting the root, . . . $y + \frac{1}{2} = \pm\frac{5}{2}$;

$$\therefore\ y = \frac{-1 \pm 5}{2} = 2, \text{ or } -3;$$

and, ... $\sqrt{2x+6}\ (=y^2) = 4$, or 9;

squaring, ... $2x + 6 = 16$, or 81;

transposing, ... $2x = 10$, or 75;

$$\therefore\ x = 5, \text{ or } 37\tfrac{1}{2}.$$

Ex. 6. Given $\dfrac{1}{(2x-4)^2} = \dfrac{1}{8} + \dfrac{2}{(2x-4)^4}$.

substituting y^2 for $\dfrac{1}{(2x-4)^4}$, we shall have, by transposing and changing

signs, ... $2y^2 - y = -\frac{1}{8}$;

$\div$ by 2, ... $y^2 - \frac{y}{2} = -\frac{1}{16}$;

completing, . $y^2 - \frac{1}{2}y + \frac{1}{16} = -\frac{1}{16} + \frac{1}{16} = 0$;

extracting the root, . . . $y - \frac{1}{4} = 0$;

$$\therefore\ y = \frac{1}{4};$$

then, ... $\frac{1}{(2x-4)^2}(=y)=\frac{1}{4}$;

extracting the root, . . . $\frac{1}{2x-4}=\pm\frac{1}{2}$;

clearing, ... $2x-4=\pm 2$;

transposing and $\div$ by 2, . . $x=2\pm 1=3$, or 1.

Ex. 7. Given $3x^{\frac{4}{3}}-\frac{5x^{\frac{8}{3}}}{2}+592=0$.

transpo. and changing signs, $\frac{5x^{\frac{8}{3}}}{2}-3x^{\frac{4}{3}}=592$;

$\times$ by $\frac{2}{5}$, ... $x^{\frac{8}{3}}-\frac{6}{5}x^{\frac{4}{3}}=\frac{1184}{5}$;

completing, $x^{\frac{8}{3}}-\frac{6}{5}x^{\frac{4}{3}}+\frac{9}{25}=\frac{1184}{5}+\frac{9}{25}=\frac{5929}{25}$;

extracting the root, $x^{\frac{4}{3}}-\frac{3}{5}=\pm\sqrt{\frac{5929}{25}}=\pm\frac{77}{5}$; . . . $(x^{\frac{4}{3}}=\sqrt{x^{\frac{8}{3}}})$

$$\therefore\ x^{\frac{4}{3}}=\frac{3\pm 77}{5}=16, \text{ or } -\frac{74}{5}.$$

$$\text{then, } \left.\begin{aligned} x^{\frac{4}{3}}&=16\\ x^{\frac{1}{3}}&=2\\ x&=8\end{aligned}\right\}\text{ or }\left\{\begin{aligned} x^{\frac{4}{3}}&=-\frac{74}{5}\\ x^{4}&=\left(-\frac{74}{5}\right)^{3}\\ x&=\left(-\frac{74}{5}\right)^{\frac{3}{4}}.\end{aligned}\right.$$

Ex. 8. Given $(x+12)^{\frac{1}{2}}=6-(x+12)^{\frac{1}{4}}$.

substituting y^2 for $(x+12)^{\frac{1}{2}}$ and y for $(x+12)^{\frac{1}{4}}$, we have, by transposing, ... $y^2+y=6$;

completing, . . . $y^2 + y + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$;

extracting the root, $y + \frac{1}{2} = \pm\sqrt{\frac{25}{4}} = \pm\frac{5}{2}$;

$\therefore\ y = \frac{-1 \pm 5}{2} = 2$, or -3;

and, $(x+12)^{\frac{1}{2}} (= y^2) = 4$, or 9;

squaring, ... $x + 12 = 16$, or 81;

transposing, ... $x = 4$, or 69.

Ex. 9. Given $x = \frac{\sqrt{x^4 - a^4}}{a}$.

clearing, ... $\sqrt{x^4 - a^4} = ax$;

squaring, ... $x^4 - a^4 = a^2 x^2$;

transpo. and compl., $x^4 - a^2 x^2 + \frac{a^4}{4} = a^4 + \frac{a^4}{4} = \frac{5a^4}{4}$;

extracting the root, $x^2 - \frac{a^2}{2} = \pm\sqrt{\frac{5a^4}{4}} = \pm\frac{a^2\sqrt{5}}{2}$;

transposing, ... $x^2 = \frac{a^2 \pm a^2\sqrt{5}}{2}$;

extracting the root, $x = \pm\sqrt{\frac{a^2 \pm a^2\sqrt{5}}{2}} = \pm a\sqrt{\frac{1 \pm \sqrt{5}}{2}}$.

Ex. 10. Given $x^{\frac{5}{2}} - x = 56x^{-\frac{1}{2}}$, or $\frac{56}{x^{\frac{1}{2}}}$. (Alg. p. 38.)

$\times$ by $x^{\frac{1}{2}}$, ... $x^{\frac{6}{2}} - x^{\frac{3}{2}} = 56$; . . . . . $(x = x^{\frac{2}{2}})$

completing, . . $x^{\frac{6}{2}} - x^{\frac{3}{2}} + \frac{1}{4} = 56 + \frac{1}{4} = \frac{225}{4}$;

extracting the root, $x^{\frac{3}{2}} - \frac{1}{2} = \pm\sqrt{\frac{225}{4}} = \pm\frac{15}{2}$; . . . $(x^{\frac{3}{2}} = \sqrt{x^{\frac{6}{2}}})$

$$\therefore \quad x^{\frac{3}{2}} = \frac{1 \pm 15}{2} = 8, \text{ or } -7.$$

Then, $\quad x^{\frac{3}{2}} = 8, \text{ or } -7;$

$$x^3 = 64, \text{ or } 49;$$

$$x = 4, \text{ or } \sqrt[3]{49}.$$

Ex. 11. Given $3x^2 + x^{\frac{7}{6}} - 3104\, x^{\frac{1}{3}} = 0.$

$\div$ by $x^{\frac{1}{3}}$, ... $3x^{\frac{5}{3}} + x^{\frac{5}{6}} - 3104 = 0$; . . . . $(x^2 = x^{\frac{6}{3}})$

transposing and $\div$ by 3, $\quad x^{\frac{5}{3}} + \frac{x^{\frac{5}{6}}}{3} = \frac{3104}{3}$;

completing, . . $x^{\frac{5}{3}} + \frac{x^{\frac{5}{6}}}{3} + \frac{1}{36} = \frac{3104}{3} + \frac{1}{36} = \frac{37249}{36}$;

extracting the root, $x^{\frac{5}{6}} + \frac{1}{6} = \pm\sqrt{\frac{37249}{36}} = \pm\frac{193}{6}$; . $(x^{\frac{5}{6}} = \sqrt{x^{\frac{5}{3}}})$

$$\therefore \quad x^{\frac{5}{6}} = \frac{-1 \pm 193}{6} = 32, \text{ or } -32\tfrac{1}{3}.$$

Then, $\quad x^{\frac{5}{6}} = \quad 32, \text{ or } -32\tfrac{1}{3} = -\frac{97}{3};$

$$x^{\frac{1}{6}} = \quad 2, \text{ or } \left(-\frac{97}{3}\right)^{\frac{1}{5}}$$

$$x = (2)^6 = 64, \text{ or } \left(-\frac{97}{3}\right)^{\frac{6}{5}}.$$

Ex. 12. Given $[(2x + 1)^2 + x]^2 - x = 90 + (2x + 1)^2.$

transposing, $[(2x + 1)^2 + x]^2, = 90 + (2x + 1)^2 + x$;

substituting y^2 for $[(2x + 1)^2 + x]^2$, and y for $(2x + 1)^2 + x$; we have,

by transposition, . . . $y^2 - y = 90$;

completing, . . . $y^2 - y + \frac{1}{4} = 90 + \frac{1}{4} = \frac{361}{4}$;

extracting the root, . . $y - \frac{1}{2} = \pm \frac{19}{2}$;

$$\therefore\ y = \frac{1 \pm 19}{2} = 10, \text{ or } -9.$$

that is, $(2x+1)^2 + x\ (=y) = 10$, or -9;

or, $4x^2 + 4x + 1 + x = 10$, or -9;

transposing, . . . $4x^2 + 5x = 9$, or -10;

$\div$ by 4, and completing,

$$x^2 + \frac{5x}{4} + \frac{25}{64} = \left(\frac{9}{4} + \frac{25}{64} = \frac{169}{64}\right), \text{ or } \left(-\frac{5}{2} + \frac{25}{64} = -\frac{135}{64}\right);$$

extracting the root, $x + \frac{5}{8} = \pm \frac{13}{8}$, or $\pm \frac{\sqrt{-135}}{8}$;

$$\therefore\ x = \frac{-5 \pm 13}{8} = 1, \text{ or } -2\tfrac{1}{4}; \text{ or } x = \frac{-5 \pm \sqrt{-135}}{8}.$$

QUESTIONS PRODUCING QUADRATIC EQUATIONS INVOLVING BUT ONE UNKNOWN QUANTITY.

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Ex. 7. Let x be the less, and the greater will be $14 - x$.

Then, $4x^2 = 18\ (14 - x)$;

transposing, ... $4x^2 + 18x = 252$;

$\div$ by 4, and compl., $x^2 + \frac{18}{4}x + \frac{81}{16} = 63 + \frac{81}{16} = \frac{1089}{16}$;

extracting the root, $x + \frac{9}{4} = \pm\sqrt{\frac{1089}{16}} = \pm\frac{33}{4}$;

$\therefore$ the positive value of x, is $x = \frac{-9+33}{4} = 6$,

and $14 - x = 8$.

Ex. 8. Let x be one part, the other will be $48 - x$.

then, by question, . . $x(48 - x) = 432$;

ch. signs, and compl. $x^2 - 48x + (24)^2 = -432 + 576 = 144$;

extracting the root, . . $x - 24 = \pm 12$;

$\therefore x = 36$, or 12.

and consequently, $48 - x = 12$ or 36.

Ex. 9. Let x be one part, and $24 - x$ will be the other.

Then by question, $x(24 - x) = 35\{x - (24 - x)\}$;

or, . . . $-x^2 + 24x = 35x - 840 + 35x$;

transpo. and ch. signs, $x^2 + 46x = 840$;

completing, $x^2 + 46x + (23)^2 = 840 + 529 = 1369$;

extracting the root, . . $x + 23 = \pm 37$;

$\therefore x = 14$,

and $24 - x = 10$.

Ex. 10. Let x be the number.

Then, by question, . . $x - \sqrt{x} = 48\frac{3}{4}$;

completing, ... $x - \sqrt{x} + \frac{1}{4} = 49$;

extracting the root, . . . $\sqrt{x} - \frac{1}{2} = \pm 7$;

$\therefore \sqrt{x} = \frac{1}{2} \pm 7 = \frac{15}{2}$, or $-\frac{13}{2}$;

squaring, . . $x = \frac{225}{4} = 56\frac{1}{4}$, or $\frac{169}{4} = 42\frac{1}{4}$.

Ex. 11. Let x be the second, then by reversing the proportion, we have

$$16 : x :: x : \frac{x^2}{16} = \text{the first number};$$

and, by question, . . $\frac{x^4}{256} + x^2 = 225$;

clearing, ... $x^4 + 256x^2 = 57600$;

completing, . . $x^4 + 256x^2 + (128)^2 = 73984$;

extracting the root, . $x^2 + 128 = \pm 272$;

transposing, . . . $x^2 = 144$, or -400;

extracting the root, . $x = 12$, or $\sqrt{-400}$;

$$\text{hence, } \frac{x^2}{16} = \frac{144}{16} = 9.$$

Ex. 12. Let x be the number, then $\frac{72}{x}\,l$ will be the price of each. Now, increasing the number by 6, and diminishing the price of each by $1l$, should produce the same sum, therefore, we shall have

$$(x+6)\left(\frac{72}{x} - 1\right) = 72:$$

that is, ... $72 - x + \frac{432}{x} - 6 = 72$;

clear. and transpo., $-x^2 - 6x = -432$;

ch. signs, and compl., $x^2 + 6x + 9 = 432 + 9 = 441$;

extracting the root, . . . $x + 3 = \pm 21$;

$$\therefore x = 18,$$

$$\text{and } \frac{72}{x} = 4.$$

Ex. 13. Let the cost price be x pounds, and the gain will be $39 - x$;

then, $100 : x :: x : 39 - x$;

$\times$ extremes and means, $x^2 = 3900 - 100x$;

transpo. and compl., $x^2 + 100x + (50)^2 = 6400$;

extracting the root, . . $x + 50 = \pm 80$;

consequently, $x = 30$.

Ex. 14. Let x be one of the numbers of coins, and the other will be $24 - x$; then, as each coin is worth as many pence as there are coins of the other kind, we have

$$x(24 - x) + (24 - x)x = 18s., \text{ or } 216 \text{ pence};$$

$\div$ by 2, and ch. signs, $x^2 - 24x = -108$;

completing, ... $x^2 - 24x + 144 = 36$;

extracting the root, . . . $x - 12 = 6$;

$\therefore x = 18$;

then, $24 - x = 6$.

Ex. 15. Suppose B (second traveller) travels x days, then A (first traveller) will have travelled $x + 5$ days; hence, the distance gone by A will be $(x + 6)\dfrac{(x + 5)}{2}$; (*Art.* 71, *Theor.* 5.), and the distance gone by B will be $12x$ miles; but as they have both reached the same point, we shall have

$$(x + 6)\frac{x + 5}{2} = 12x;$$

or, clearing, . . . $x^2 + 11x + 30 = 24x$;

transpo. and compl., . $x^2 - 13x + \left(\frac{13}{2}\right)^2 = -30 + \frac{169}{4} = \frac{49}{4}$;

extracting the root, . . $x - \frac{13}{2} = \pm \frac{7}{2}$;

$$\therefore x = \frac{13 \pm 7}{2} = 10, \text{ or } 3.$$

This result shows that B will come up to A after travelling 3 days, or 36 miles; he will then pass him, and keep in advance till the 10th day, when he will be overtaken by A, who will thenceforth continue before him.

Ex. 16. Suppose $2x$ one number.

By question, . . $2 : 3 :: 2x : 3x =$ the other number.

Sum $\times$ by product, $5x \times 6x^2 = 12\,(9x^2 - 4x^2)$, or $60x^2$;

that is, $30x^3 = 60x^2$;

$\div$ by $30x^2$, ... $x = 2$.

$$\therefore 2x = 4, \text{ and } 3x = 6.$$

Ex. 17. Let $x =$ the left-hand digit.

Then, since $\frac{1}{5}$ of his age $= x^2$, we have

$5x^2 =$ his age;

and, as the age is equal to 10 times the sum of the two digits, we have

$5x^2 - 10x =$ the right-hand digit;

hence, the two digits are x, and $5x^2 - 10x$, whose sum is $5x^2 - 9x$;

and by question, $10\,(5x^2 - 9x)$, that is, $50x^2 - 90x = 5x^2$;

$\div$ by x, and transposing, $45x = 90$;

$$\therefore x = 2;$$

whence $5x^2 - 10x = 0$;

consequently, the number is 20.

Ex. 18. Let $x =$ A's capital,

Then $26 - x =$ A's gain,

and $x - 8 =$ B's gain. . . $[= 18 - (26 - x)]$.

$\therefore\ 12x : 480 :: 26 - x : x - 8$; . . $(30 \times 16 = 480)$

$\div$ 1st and 2d terms by 12, $x : 40 :: 26 - x : x - 8$, (Art. 68.)

$\times$ extremes and means, $x^2 - 8x = 1040 - 40x$;

transpo. and compl., $x^2 + 32x + (16)^2 = 1296$;

extracting the root, . . $x + 16 = \pm 36$;

$\therefore\ x = 20$.

Ex. 19. Suppose A's stock was x pounds, then B's was £$416 - x$; also A's gain, $228 - x$, and B's, $252 - (416 - x) = x - 164$.

$\therefore\ 9x : 2496 - 6x :: 228 - x : x - 164$;

$\div$ 1st and 2d terms by 3, $3x : 832 - 2x :: 228 - x : x - 164$;

$\times$ extremes and means, $3x^2 - 492x = 189696 - 1288x + 2x^2$;

transpo. and compl., $x^2 + 796x + (398)^2 = 348100$;

extracting the root, . . $x + 398 = \pm 590$;

$\therefore\ x = 192$;

then, $416 - x = 224$.

Ex. 20. Let x yards be the breadth; the length will be $x + 16$.

Then, $x(x + 16) = 960$;

completing, . . . $x^2 + 16x + 64 = 1024$;

extracting the root, . $x + 8 = \pm 32$;

$\therefore\ x = 24$;

and $x + 16 = 40$.

Ex. 21. Let x inches be the width of the frame; then $18 + 2x =$ length, and $12 + 2x =$ height. But, as the frame is to be equal in surface to the glass (which is $12 \times 18 = 216$); we shall have for the whole surface of both,

$$(18 + 2x)(12 + 2x) = 432; \text{ (= twice surface of the glass)}$$

that is, $216 + 60x + 4x^2 = 432$;

transposing, ... $4x^2 + 60x = 216$;

adding $(15)^2$†, $4x^2 + 60x + (15)^2 = 216 + (15)^2 = 441$;

extracting the root, . $2x + 15 = \pm 21$;

$$\therefore x = \frac{6}{2} = 3.$$

Ex 22. Here, we will suppose the hypothenuse to be x; then, as the square of the hypothenuse is equal to the sum of the squares of the sides in a right-angled triangle, we shall have

$$x^2 = (x - 6)^2 + (x - 3)^2;$$

or $x^2 = 2x^2 - 18x + 45$;

transpo. and ch. signs, $x^2 - 18x = -45$;

completing, . . . $x^2 - 18x + 81 = 36$;

extracting the root, $x - 9 = \pm 6$;

$$\therefore x = 15.$$

Consequently, $x - 6 = 9$, and $x - 3 = 12$.

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† The student will remark that the first two terms of this equation are the same as would have arisen from the equation $x^2 + 15x = 54$, being multiplied by 4, according to the method taken from the Bija Ganita (see Alg. p. 108.); consequently, the addition of $(15)^2$ completes the square.

Ex. 23. Here, let x = the first number.

$$15 - x = \text{second},$$

$$15 - x + 36 = \text{third};$$

then, since the product of the extremes = the square of the mean, we have

$$x(15 - x + 36) = (15 - x)^2;$$

or, $\quad -x^2 + 51x = 225 - 30x + x^2;$

transpo. and ch. signs, $\quad 2x^2 - 81x = -225;$

$\div$ by 2, and compl., $x^2 - \frac{81}{2}x + \left(\frac{81}{4}\right)^2 = -\frac{225}{2} + \left(\frac{81}{4}\right)^2 = \frac{4761}{16};$

extracting the root, $\quad x - \frac{81}{4} = \pm\frac{69}{4};$

$$\therefore x = \frac{81 \pm 69}{4} = 3, \text{ or } 37\tfrac{1}{2}.$$

Consequently, $15 - x = 12$, and $51 - x = 48$.

Ex. 24. Let x denote the number of seconds.

Then, $\quad 1^2 : x^2 :: 16\frac{1}{12} : 16\frac{1}{12}x^2$ = the whole space passed through;

and, $\quad 1^2 : (x-1)^2 :: 16\frac{1}{12} : 16\frac{1}{12}(x-1)^2$ = the space passed through previously to the last second; hence, by question

$$\frac{193}{12}x^2 - \frac{193}{12}(x^2 - 2x + 1) = 595;$$

$\times$ by $\frac{12}{193}$, . . . $x^2 - x^2 + 2x - 1 = \frac{7140}{193};$

and, by transposition, $\quad 2x = \frac{7140}{193} + 1 = \frac{7333}{193};$

$$\therefore x = \frac{7333}{386}.$$

And $16\frac{1}{12}x^2 = \frac{193}{12} \times \frac{53772889}{148996} = \frac{10378167577}{1787952} = 5804\frac{1}{2}$ feet.

ON QUADRATICS INVOLVING TWO UNKNOWN QUANTITIES.

When one of the given Equations is in the form of a Simple Equation.

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Ex. 6. Given $\begin{cases} \dfrac{x}{y^2} = 2, \\ \frac{1}{3}(x-y) = 5. \end{cases}$

From the first equation we have $x = 2y^2$, and, by substitution in

the second, ... $\dfrac{2y^2-y}{3} = 5$;

clearing, ... $2y^2 - y = 15$;

$\times$ by (4×2), and compl., $16y^2 - 8y + 1 = 121$; . . . (Art. 90)

extracting the root, . . $4y - 1 = \pm 11$;

$$\therefore y = \frac{1 \pm 11}{4} = 3, \text{ or } -2\tfrac{1}{2}.$$

$$\text{And } x\,(= 2y^2) = 18, \text{ or } 12\tfrac{1}{2}.$$

Ex. 7. Given $\begin{cases} x + 4y = 14, \\ y^2 - 2y + 4x = 11. \end{cases}$

2d Eq. . . . . $y^2 - 2y + 4x = 11$

1st Eq. $\times$ 4, ... $16y + 4x = 56$

subtracting, ... $y^2 - 18y = -45$

completing, ... $y^2 - 18y + 81 = 36$;

extracting the root, ... $y - 9 = \pm 6$;

$$\therefore y = 15, \text{ or } 3.$$

$$\text{And } x\,(= 14 - 4y) = -46, \text{ or } 2.$$

Ex. 8. Given $\begin{cases} 2x + y = 22, \\ \dfrac{xy}{2} + y^2 = 60. \end{cases}$

2d Eq. $\times$ 4, ... $2xy + 4y^2 = 240$

1st Eq. $\times$ y, ... $2xy + y^2 = 22y$

subtracting, ... $3y^2 = 240 - 22y$;

transpo., and $\times$ by (4×3), $36y^2 + 264y = 2880$; . . . (Art. 90)

completing, . . . $36y^2 + 264y + (22)^2 = 3364$;

extracting the root, . . $6y + 22 = \pm 58$;

$$\therefore y = \frac{-22 \pm 58}{6} = 6, \text{ or } -13\tfrac{1}{3}.$$

$$\text{Hence, } x \left(= \frac{22 - y}{2}\right) = 8, \text{ or } 17\tfrac{2}{3}.$$

Ex. 9. Given $\begin{cases} x = 15 + y, \\ y^3 = \dfrac{xy}{2}. \end{cases}$

2d Eq. $\div$ by y, $y^2 = \dfrac{x}{2}$, or substituting, $y^2 = \dfrac{15 + y}{2}$;

clearing, and transpo., . . $2y^2 - y = 15$;

$\times$ by (4×2), and compl., $16y^2 - 8y + 1 = 121$;

extracting the root, . . $4y - 1 = \pm 11$;

$$\therefore y = 3, \text{ or } -2\tfrac{1}{2}.$$

$$\text{Consequently, } x\ (= 15 + y) = 18, \text{ or } 12\tfrac{1}{2}.$$

Ex. 10. Given $\begin{cases} x + 3y = 16, \\ 3x^2 + 2xy - y^2 = -12. \end{cases}$

From first equation, we have $x = 16 - 3y$; and substituting in the

second, ... $20y^2 - 256y + 768 = -12$;

transpo. and $\div$ by 20, $y^2 - \frac{64}{5}y = -\frac{195}{5}$;

completing, $y^2 - \frac{64}{5}y + \left(\frac{32}{5}\right)^2 = -\frac{195}{5} + \left(\frac{32}{5}\right)^2 = \frac{49}{(5)^2}$;

extracting the root, . . $y - \frac{32}{5} = \pm\frac{7}{5}$;

$$\therefore\ y = 5, \text{ or } 7\tfrac{4}{5}.$$

$$\text{And } x\ (= 16 - 3y) = 1, \text{ or } -7\tfrac{2}{5}.$$

Ex. 11. Given $\begin{cases} x + y : x - y :: 13 : 5, \\ x + y^2 = 25. \end{cases}$

$\times$ Extremes and means, $13x - 13y = 5x + 5y$;

transposing, ... $8x = 18y$;

$$\therefore\ x = \frac{9}{4}y.$$

Substituting in second, $y^2 + \frac{9}{4}y = 25$;

completing, . . $y^2 + \frac{9}{4}y + \left(\frac{9}{8}\right)^2 = \frac{1681}{(8)^2}$;

extracting the root, . . $y + \frac{9}{8} = \pm\frac{41}{8}$;

$$\therefore\ y = 4, \text{ or } -6\tfrac{1}{4}.$$

$$\text{Whence } x \left(= \frac{9}{4}y\right) = 9, \text{ or } -14\tfrac{1}{16}.$$

Ex. 12. Given $\begin{cases} \dfrac{x^2}{y^2} + \dfrac{4x}{y} = \dfrac{85}{9}, \\ x - y = 2. \end{cases}$

Here, the left side of the first equation has the form of the first two terms of a square; therefore, we have, by completing,

$$\left(\frac{x}{y}\right)^2 + 4\left(\frac{x}{y}\right) + 4 = \frac{121}{9};$$

extracting the root, . . $\frac{x}{y} + 2 = \pm\frac{11}{3};$

$$\therefore \frac{x}{y} = \frac{-6 \pm 11}{3} = \frac{5}{3}, \text{ or } -\frac{17}{3}.$$

And substituting in this the value of x, given by the second equation, that is, $x = 2 + y$, we have

$$\frac{2+y}{y} = \frac{5}{3}, \text{ or } -\frac{17}{3};$$

clear. and transpo., $\left.\begin{array}{r} 2y = 6 \\ \therefore\ y = 3 \end{array}\right\}$ or $\left\{\begin{array}{r} -20y = 6, \\ \therefore\ y = -\frac{3}{10}; \end{array}\right.$

and $x\ (= 2 + y) = 5$, or $1\frac{7}{10}$.

Particular Examples.

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Ex. 1. Let x and y be the two numbers.

Then, $\quad x + y = 24,$

$\qquad x^2 + y^2 = 306;$

1st Eq. squared, . . $x^2 + 2xy + y^2 = 576$

2d Eq. . . . . . . $x^2 \qquad + y^2 = 306$

by subtraction, . . . $2xy \qquad = 270$; and subtracting this

from 2d Eq., . . . $x^2 \qquad + y^2 = 306,$

we have, ... $x^2 - 2xy + y^2 = 36;$

extracting the root, ... $x - y = \pm 6$

1st Eq., ... $x + y = 24$

by adding, ... $2x = 30, \text{ or } 18;$

$$\therefore x = 15, \text{ or } 9;$$

by subtracting, ... $2y = 18, \text{ or } 30;$

$$\therefore y = 9, \text{ or } 15.$$

Ex. 2. Let x and y be the numbers; then we shall have

$$x + y = 27,$$

$$x^3 + y^3 = 4941;$$

and putting $x = s + z$, and $y = s - z$, we have, by adding,

$$x + y = 2s = 27;$$

also, $x^3 = s^3 + 3s^2z + 3sz^2 + z^3,$

and $y^3 = s^3 - 3s^2z + 3sz^2 - z^3;$

by adding, . $x^3 + y^3 = 2s^3 + 6sz^2 = 4941;$ . . (2d Eq.)

transposing, . . . $6sz^2 = 4941 - 2s^3;$

$$\therefore z^2 = \frac{4941 - 2s^3}{6s};$$

$$\text{that is, } z^2 = \frac{4941}{81} - \frac{729}{12} = \frac{549}{9} - \frac{243}{4} = \frac{4}{36} = \frac{1}{4}.$$

$$\text{hence, } z = \tfrac{1}{2}.$$

Consequently, x ($= s + z$, that is, $\frac{27}{2} + \frac{1}{2}$) $= 14$; and y ($= s - z$) $= 13$.

Ex. 3. Let s represent half the sum of the numbers, and z half their difference; then the numbers will be

$$s + z, \text{ and } s - z;$$

and putting $m = 2657$, we shall have

$$(s+z)^4 + (s-z)^4 = m;$$

$$\text{but } (s+z)^4 = s^4 + 4s^3z + 6s^2z^2 + 4sz^3 + z^4$$

$$\text{and } (s-z)^4 = s^4 - 4s^3z + 6s^2z^2 - 4sz^3 + z^4$$

$$\therefore \text{ adding}, \quad 2s^4 \quad + 12s^2z^2 \quad + 2z^4 = m,$$

transposing, . . . $2z^4 + 12s^2z^2 = m - 2s^4$;

$$\text{or}, \quad z^4 + 6s^2z^2 = \frac{m}{2} - s^4;$$

completing, . . $z^4 + 6s^2z^2 + 9s^4 = \frac{m}{2} - s^4 + 9s^4$;

extracting the root,

$$z^2 + 3s^2 = \pm\sqrt{\frac{m}{2} - s^4 + 9s^4} = \pm\sqrt{\frac{m}{2} + 8s^4};$$

transpo. and extr. root, $z = \sqrt{-3s^2 \pm \sqrt{\frac{m}{2} + 8s^4}}$;

but, by restoring the values of $s = \frac{11}{2}$, and $m = 2657$, we have

$$z = \sqrt{-3s^2 \pm \sqrt{\frac{m}{2} + 8s^4}} = \sqrt{-\frac{363}{4} \pm 93} = \sqrt{\frac{9}{4}} = \pm\frac{3}{2};$$

$$\text{hence,} \quad s + z\left(=\frac{11+3}{2}\right) = 7, \text{ and } s - z\left(=\frac{11-3}{2}\right) = 4.$$

Ex. 4. Let the numbers be $s+z$ and $s-z$;

and putting $m = 17050$, we shall have

$$(s+z)^5 + (s-z)^5 = m:$$

now, developing the powers in the first side, we have

$$(s+z)^5 = s^5 + 5s^4z + 10s^3z^2 + 10s^2z^3 + 5sz^4 + z^5$$
$$(s-z)^5 = s^5 - 5s^4z + 10s^3z^2 - 10s^2z^3 + 5sz^4 - z^5$$

$\therefore$ adding, . $2s^5 \qquad + 20s^3z^2 \qquad + 10sz^4 = m$;

transposing, . . . $10sz^4 + 20s^3z^2 = m \quad - 2s^5$;

$$\text{or, } z^4 + \ 2s^2z^2 = \frac{m}{10s} - \frac{2s^4}{10};$$

completing, . . . $z^4 + 2s^2z^2 + s^4 = \dfrac{m}{10s} - \dfrac{s^4}{5} + s^4$;

extracting the root, $z^2 + s^2 = \pm\sqrt{\dfrac{m}{10s} - \dfrac{s^4}{5} + s^4}$;

transposing and extracting root,

$$z = \sqrt{-s^2 \pm \sqrt{\frac{m}{10s} - \frac{s^4}{5} + s^4}} = \sqrt{-25 \pm 29} = \sqrt{4} = 2.\ .\ (s=5)$$

hence, $s + z = 7$, and $s - z = 3$.

Ex. 5. Let x and y be the two numbers.

$$\text{Then, } x + y = 47 = s;$$
$$xy = 546 = p;$$

squaring 1st Eq. . . $x^2 + 2xy + y^2 = s^2$;

2d Eq. $\times$ 2, . . . $2xy \qquad = 2p$;

subtracting, ... $x^2 \qquad + y^2 = s^2 - 2p$;

that is, $x^2 + y^2 = 2209 - 1092 = 1117$.

Ex. 6. Let x and y be the numbers.

$$\text{Then, } x + y = 20 = s;$$
$$xy = 99 = p;$$

squaring 1st Eq., . . $x^2 + 2xy + y^2 = s^2$;

2d Eq. × 2, ... $2xy \quad = 2p$;

Subtracting, ... $x^2 \quad + y^2 = s^2 - 2p$; . . . (A)

× by $(x + y)$, . $x^3 + x^2y + xy^2 + y^3 = (s^2 - 2p)\, s$; . . $(s = x + y)$

or, $x^3 + ps + y^3 = s^3 - 2ps$; . $(ps = x^2y + xy^2)$

transposing, ... $x^3 + y^3 = s^3 - 3ps$;

that is, $x^3 + y^3 = 8000 - 5940 = 2060$.

Ex. 7. Let the numbers be x and y.

Then, $x + y = 19 = s$;

$xy = 78 = p$.

By proceeding as in last example, we find

$$x^3 + y^3 = s^3 - 3ps;$$

× by $(x + y)$, . $x^4 + x^3y + xy^3 + y^4 = (s^3 - 3ps)\, s$; . $(s = x + y)$

*that is, $x^4 + p\,(s^2 - 2p) + y^4 = s^4 - 3ps^2$;

and transposing, . . $x^4 + y^4 = s^4 - 4ps^2 + 2p^2$.

that is, $x^4 + y^4 = 130321 - 112632 + 12168 = 29857$.

---

* Here we have substituted $p\,(s^2 - 2p)$ for $x^3y + xy^3$, but the learner will easily perceive that these two quantities are equal; for, in the preceding Example, we have by the equation (A)

$$x^2 + y^2 = s^2 - 2p;$$

and, $xy \quad = p$

hence, by multiplication, $x^3y + xy^3 = p\,(s^2 - 2p)$

When both Equations have a Quadratic form.

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Ex. 3. Given $\begin{cases} x^2 + xy = 12, \\ xy - 2y^2 = 1. \end{cases}$

By substituting zy for x, the equations become

$$z^2y^2 + zy^2, \text{ or } (z^2 + z)\, y^2 = 12;$$

$$zy^2 - 2y^2, \text{ or } (z - 2)\, y^2 = 1;$$

from the first equation we have

$$y^2 = \frac{12}{z^2 + z};$$

and from the second,

$$y^2 = \frac{1}{z - 2}; \quad . \quad . \quad . \quad (A)$$

$$\text{consequently, } \frac{12}{z^2 + z} = \frac{1}{z - 2};$$

clearing, ... $^*12z - 24 = z^2 + z;$

transpo. and ch. signs, . $z^2 - 11z = -24;$

completing, . . $z^2 - 11z + \left(\frac{11}{2}\right)^2 = \frac{25}{4};$

extracting the root, . . $z - \frac{11}{2} = \pm \frac{5}{2};$

$$\therefore\ z = 8, \text{ or } 3.$$

$$\text{Hence (A), } y^2 \left(= \frac{1}{z - 2}\right) = \frac{1}{6}, \text{ or } 1;$$

---

* Or, this value may be obtained immediately by dividing one of the preceding equations by the other, thus: $\frac{(z^2 + z)\ y^2}{(z - 2)\ y^2} = \frac{12}{1}$, that is, $\frac{z^2 + z}{z - 2} = 12$, and then clearing. This method will also apply to the two following examples.

$$\therefore y = \pm \frac{1}{\sqrt{6}}, \text{ or } \pm 1.$$

$$\text{and } x\ (= zy) = \pm \frac{8}{\sqrt{6}}, \text{ or } \pm 3.$$

Ex. 4. Given $\begin{cases} 3x^2 + xy = 68, \\ 4y^2 + 3xy = 160. \end{cases}$

Let $x = zy$, and the equations will be

$$3z^2y^2 + zy^2, \text{ or } (3z^2 + z)\, y^2 = 68;$$

$$4y^2 + 3zy^2, \text{ or } (4 + 3z)\, y^2 = 160;$$

from the first of these we obtain

$$y^2 = \frac{68}{3z^2 + z}; \quad . \quad . \quad . \quad (A)$$

and from the second,

$$y^2 = \frac{160}{4 + 3z}; \quad . \quad . \quad . \quad (B)$$

hence, $\div$ by 4, . . . $\dfrac{17}{3z^2 + z} = \dfrac{40}{4 + 3z}$;

clearing, ... $68 + 51z = 120z^2 + 40z$;

transpo. and ch. signs, $120z^2 - 11z = 68$;

$\times$ by 4 times coeff., $57600z^2 - 5280z = 32640$;

completing, . . $57600z^2 - 5280z + 121 = 32761$;

extracting the root, . $240z - 11 = \pm 181$;

$$\therefore z = \frac{11 \pm 181}{240} = \frac{192}{240}, \text{ or } -\frac{170}{240}, \text{ that is } \frac{4}{5}, \text{ or } -\frac{17}{24}.$$

whence (Eq. B), $y \left(= \sqrt{\dfrac{160}{4 + 3z}}\right) = \pm 5, \text{ or } \pm \sqrt{\dfrac{256}{3}} = \pm \dfrac{\sqrt{768}}{3}$;

(which latter is obtained by $\times$ num. and den. of preceding fraction by 3)

$$\text{and } x\ (= zy) = \pm 4, \text{ or } \mp \frac{17\sqrt{768}}{72}.$$

Ex. 5. Given $\begin{cases} 2x^2 - 3xy + y^2 = 4, \\ 2xy - 3y^2 - x^2 = -9. \end{cases}$

By putting zy for x in these equations, we have

$$2z^2y^2 - 3zy^2 + y^2, \text{ or } (2z^2 - 3z + 1)\,y^2 = 4;$$

$$2zy^2 - 3y^2 - z^2y^2, \text{ or } (2z - 3 - z^2)\,y^2 = -9;$$

from the first of which we obtain

$$y^2 = \frac{4}{2z^2 - 3z + 1}; \quad . \quad . \quad . \quad (A)$$

and from the second,

$$y^2 = \frac{-9}{2z - 3 - z^2};$$

$$\therefore \frac{4}{2z^2 - 3z + 1} = \frac{-9}{2z - 3 - z^2};$$

clearing, . . $8z - 12 - 4z^2 = -18z^2 + 27z - 9$;

transposing, ... $14z^2 - 19z = 3$;

$\times$ 4 times coeff., . $784z^2 - 1064z = 168$;

completing, . . $784z^2 - 1064z + 361 = 529$;

extracting the root, . . $28z - 19 = \pm 23$;

$$\therefore z = \frac{19 \pm 23}{28} = \frac{3}{2}, \text{ or } -\frac{1}{7}:$$

whence (Eq. A), $y \left(= \sqrt{\dfrac{4}{2z^2 - 3z + 1}}\right) = \pm 2$, or $\sqrt{\dfrac{49}{18}} = \pm \dfrac{7}{\sqrt{18}}$;

and $x\ (= zy) = \pm \dfrac{6}{2} = \pm 3$; or $-\dfrac{1}{7} \times \pm \dfrac{7}{\sqrt{18}} = \mp \dfrac{1}{\sqrt{18}}$.

MISCELLANEOUS EXAMPLES,

To which the preceding Methods do not immediately apply.

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Ex. 6. Given $\begin{cases} x^2 + x = \dfrac{12}{y}, \\ x^3y + y = 18. \end{cases}$

$$\text{or,}\quad x(x+1)=\frac{12}{y};$$

$$x^3+1=\frac{18}{y}.$$

consequently, we have $\dfrac{x^3+1}{x(x+1)}=\dfrac{18}{12}=\dfrac{3}{2}$;* that is, $\dfrac{x^2-x+1}{x}=\dfrac{3}{2}$;

then, $\times$ by x, ... $x^2-x+1=\dfrac{3}{2}x$;

or, transposing, ... $x^2-\dfrac{5}{2}x=-1$;

and completing, . . $x^2-\dfrac{5}{2}x+\left(\dfrac{5}{4}\right)^2=\dfrac{9}{16}$;

extracting the root, . . $x-\dfrac{5}{4}=\pm\dfrac{3}{4}$;

$$\text{whence,}\quad x=2,\ \text{or}\ \frac{1}{2}.$$

And, substituting in the second equation, we have

$(2^3=x^3)$ . . . $8y+y$, that is, $9y=18$; $\therefore\ y=2$; or

$\left(\dfrac{1}{2}\right)^3=x^3)$, . $\dfrac{1}{8}y+y=\dfrac{9}{8}y=18$, and $9y=144$; $\therefore y=16$.

Ex. 7. Given $\begin{cases} x^2+4y^2=256-4xy, \\ 4y^2-x^2=\ 64. \end{cases}$

1st Eq. transpo. . . $x^2+4yx+4y^2=256$;

extracting the root, . . $x+2y=\pm 16$;

$$\therefore\ x=\pm 16-2y.$$

---

* The learner will recollect (Art. 34,) that $\dfrac{\frac{18}{y}}{\frac{12}{y}}=\dfrac{18}{y}\times\dfrac{y}{12}=\dfrac{18y}{12y}$ $=\dfrac{18}{12}=\dfrac{3}{2}$; and also, that the numerator and denominator of the fraction $\dfrac{x^3+1}{x(x+1)}$, divided by $x+1$, will become $\dfrac{x^2-x+1}{x}$.

And substituting this value in the second equation, we have

$$4y^2 - (\pm 16 - 2y)^2 = 64;$$

$$\text{that is, } \pm 64y - 256 = 64;$$

$$\text{or, } \pm 64y = 320;$$

$$\therefore y = \pm 5;$$

$$\text{hence, } x\ (= \pm 16 - 2y) = \pm 6.$$

Ex. 8. Given $\begin{cases} (x^2 + y^2) \times (x - y) = 51, \\ x^2 + y^2 + x = 20 + y. \end{cases}$

From 1st Eq. . . . $x^2 + y^2 = \dfrac{51}{x - y}$; . . . (A)

and from 2d, . . . $x^2 + y^2 = 20 + y - x$.

Hence, equating these two values of $x^2 + y^2$, we have

$$\frac{51}{x - y} = 20 + y - x, \text{ or } 20 - (x - y);$$

clearing, ... $51 = 20\,(x - y) - (x - y)^2$;

$$\text{or, } (x - y)^2 - 20\,(x - y) = -51;$$

completing, . $(x - y)^2 - 20\,(x - y) + 100 = 49$;

extracting the root, . $(x - y) - 10 = \pm 7$;

$$\therefore x - y = 17, \text{ or } 3 \quad . \; . \; . \quad \text{(B)}$$

Substituting these values of $x - y$ in (A), we have

$$x^2 + y^2 = \quad 3, \text{ or } 17; \; . \; . \; . \; \text{(C)}$$

Eq. (B) squared . . $x^2 - 2xy + y^2 = 289, \text{ or } 9;$

subtracting, . . . $\quad 2xy \quad = -286, \text{ or } \quad 8;$

adding Eq. (C), . . $\quad x^2 + y^2 = \quad 3, \text{ or } 17$

$$x^2 + 2xy + y^2 = -283, \text{ or } 25;$$

extracting the root, . $x+y=\pm\sqrt{-283}$, or ± 5.

But, Eq. (B) . . $x-y=\quad 17,$ or $\quad 3$

$\therefore$ adding . . . . $2x \quad =17\pm\sqrt{-283}$; or $\quad 8$, or -2.

$$\therefore x=\frac{17\pm\sqrt{-283}}{2};\ \text{or } 4, \text{ or } -1;$$

and subtracting the same, $2y=-17\pm\sqrt{-283}$; or 2, or -8.

$$\therefore y=\frac{-17\pm\sqrt{-283}}{2};\ \text{or } 1, \text{ or } -4.$$

QUESTIONS PRODUCING QUADRATIC EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES.

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Ex. 7. Let x be the greater, and y the less.

$$\text{Then, } (x+y)\,x=144;$$
$$(x-y)\,y=\ 14;$$

by substituting zy for x, the equations become changed into

$$z^2y^2+zy^2, \text{ or } (z^2+z)\,y^2=144;$$
$$zy^2-\ y^2, \text{ or } (z-1)\,y^2=\ 14;$$

from the first of which, . $y^2=\dfrac{144}{z^2+z}$;

and from the second, . . $y^2=\dfrac{14}{z-1}$; . . . . (A)

Consequently, $\dfrac{144}{z^2+z}=\dfrac{14}{z-1}$;

clearing, ... $144z-144=14z^2+14z$,

transpo. and ch. signs, $14z^2-130z=-144$;

or, $\div$ by 2, . . . $7z^2 - 65z = -72$;

$\times$ by (4×7), . . $196z^2 - 1820z = -2016$;

completing, . . . $196z^2 - 1820z + (65)^2 = 2209$;

extracting the root, . $14z - 65 = \pm 47$;

$$z = \frac{65 \pm 47}{14} = 8, \text{ or } \frac{9}{7}.$$

Hence (A), $y^2 \left(= \frac{14}{z - 1}\right) = 2$, or 49;

$$\therefore\ y = \sqrt{2}, \text{ or } \pm 7;$$

and $x\ (= zy) = 8\sqrt{2}$, or ± 9.

Ex. 8. Let x be the left-hand digit, and y the other, then, by question, we have $\frac{10x + y}{xy} = 2$, and $10x + v + 27 = 10y + x$.

Clearing 1st Eq. . . . $10x + y = 2xy$;

transpo. in 2d Eq. . . $9x - 9y = -27$,

$$\therefore\ x = \frac{9y - 27}{9} = y - 3.$$

Substituting this value in the first equation, we have

$$10y - 30 + y = 2y^2 - 6y;$$

transpo. and ch. signs, $2y^2 - 17y = -30$;

$\times$ by (4×2), . . $16y^2 - 136y = -240$;

completing, . . $16y^2 - 136y + (17)^2 = 49$;

extracting the root, . . $4y - 17 = \pm 7$;

$$\therefore\ y = 6, \text{ or } \frac{5}{2}:$$

and $x\ (= y - 3) = 3$;

consequently, the number is 36.

Ex. 9. Let x shillings be the price of the mace per lb.,

and y .... cloves ....

Then, $80x + 100y = 1300$ shill.........(A)

also, for £20 or 400 shill. he sells $\frac{400}{y}$ lb. of cloves,

and for £10 or 200*s*. .... $\frac{200}{x}$ lb. of mace;

$\therefore$ by question, . . $\frac{400}{y} - 60 = \frac{200}{x}$; . . . (B)

Eq. (A) $\div$ by 20, . . . $4x + 5y = 65$;

$$\therefore x = \frac{65 - 5y}{4}.$$

Eq. (B) $\div$ by 20, . . $\frac{20}{y} - 3 = \frac{10}{x}$;

and, by clearing, . . . $20x - 3xy = 10y$;

or, substituting for x the value found above, we have

$$325 - 25y - \frac{195y - 15y^2}{4} = 10y;$$

clear. and transpo. . . $15y^2 - 335y = -1300$;

or, $\div$ by 5, ... $3y^2 - 67y = -260$;

$\times$ by (4×3), . . $36y^2 - 804y + (67)^2 = 1369$;

extracting the root, . . $6y - 67 = \pm 37$;

$$\therefore y = \frac{67 \pm 37}{6} = 17\tfrac{1}{3}, \text{ or } 5.$$

$$\text{Whence } x \left(=\frac{65-5y}{4}\right)=10.$$

Ex. 10. Let x be the first and smallest number;

and y...... second ....

then $38-x-y$ will be the third.

But by question, $y-x-7=38-x-y-y$; . . . (A)

and $x^2+y^2+(38-x-y)^2=634$; . . . (B)

transposing Eq. (A), . . . $3y=45$;

$\therefore\ y=15.$

Substituting this value in Equation (B), we have

$$x^2+225+(23-x)^2=634;$$

$$\text{or, } x^2+225+529-46x+x^2=634;$$

transpo. and $\div$ by 2, . . $x^2-23x=-60$;

completing (Art. 90), $4x^2-92x+(23)^2=289$;

extracting the root, . . $2x-23=\pm 17$;

$\therefore\ x=20$, or 3.

And, $38-x-y=3$, or 20.

$\therefore$ the numbers are 3, 15, and 20.

Ex. 11. Let the numbers be $\frac{x}{y}$, x and xy, then, by the question,

$$\frac{x}{y}+x+xy=52,$$

$$\text{and } \frac{x}{y}+xy : x :: 10 : 3;$$

$\times$ Extr. and means, . . . $3\dfrac{x}{y} + 3xy = 10x$; . . . (A)

1st Eq. $\times$ 3, ... $3\dfrac{x}{y} + 3x + 3xy = 156$;

subtracting, . . . $-3x = 10x - 156$;

transpo. and ch. signs, . . $13x = 156$;

$\therefore\ x = 12$;

and by substituting this value of x in Eq. (A), we have

$$\frac{36}{y} + 36y = 120;$$

clearing, ... $36 + 36y^2 = 120y$;

transpo. and adding *100, $36y^2 - 120y + 100 = 64$;

extracting the root, . . $6y - 10 = \pm 8$;

$$\therefore\ y = \frac{10 \pm 8}{6} = 3, \text{ or } \tfrac{1}{3}.$$

$$\text{Hence, } \frac{x}{y} = 4,\ x = 12, \text{ and } xy = 36.$$

Ex. 12. Let the numbers be x and xy.

Then, $x^2y = x^2y^2 - x^2$;

$x^2y^2 + x^2 = x^3y^3 - x^3$;

1st Eq. $\div$ by x^2, . . . $y = y^2 - 1$; by transposing, changing signs,

and compl. . . . . $y^2 - y + \frac{1}{4} = \frac{5}{4}$;

---

* The addition of $(10)^2$ completes the square, since the other terms are the same as would have arisen from $3y^2 - 10y = -3$, multiplied by (4×3); see note, page 87 of this Key.

extracting the root, . . $y - \frac{1}{2} = \pm \frac{\sqrt{5}}{2}$;

$$\therefore y = \frac{1 \pm \sqrt{5}}{2}.$$

2d Eq. $\div$ by x^2, ... $y^2 + 1 = xy^3 - x$;

and, by substituting the preceding value of y in this equation, we have

$$\frac{1 \pm 2\sqrt{5} + 5}{4} + 1 = \frac{6 \pm 8\sqrt{5} + 10}{8} x - x;$$

that is, $$\frac{6 \pm 2\sqrt{5}}{4} + 1 = \frac{16 \pm 8\sqrt{5}}{8} x - x;$$

$$\text{or, } \frac{3 \pm \sqrt{5}}{2} + 1 = (2 \pm \sqrt{5})\, x - x;$$

$$\text{or, } \frac{5 \pm \sqrt{5}}{2} = (2 \pm \sqrt{5} - 1)\, x;$$

$$\text{or, } \frac{5 \pm \sqrt{5}}{2} = (1 \pm \sqrt{5})\, x;$$

$$\therefore x = \frac{5 \pm \sqrt{5}}{2(1 \pm \sqrt{5})}.$$

$\left\{\begin{array}{c}\times \text{ num. and} \\ \text{denom. by} \\ (1 \mp \sqrt{5})\end{array}\right\}$, . . and, $\frac{5 \pm \sqrt{5}}{2(1 \pm \sqrt{5})} = \frac{5 \pm \sqrt{5} \mp 5\sqrt{5} - 5}{-8}$

$$= \frac{\mp 4\sqrt{5}}{-8} = \pm \frac{\sqrt{5}}{2}.$$

$$\therefore xy \left(= \frac{\pm\sqrt{5}}{2} \times \frac{1 \pm \sqrt{5}}{2}\right) = \frac{5 \pm \sqrt{5}}{4}.$$

Ex. 13. Let the numbers be $x - 2y$, $x - y$, x, $x + y$, $x + 2y$.

Then, $x - 2y + x - y + x + x + y + x + 2y$, or $5x = 35$.

$$\therefore x = 7;$$

and by the question, $(x - 2y)\,(x - y)\, x\,(x + y)\,(x + 2y)$, that is,

$$x^5 - 5x^3y^2 + 4xy^4 = 10395;$$

or, by substituting the above value of x,

$$16807 - 1715y^2 + 28y^4 = 10395;$$

transpo. and $\div$ by 7, $4y^4 - 245y^2 = -916$;

completing (Art. 90), $64y^4 - 3920y^2 + (245)^2 = 45369$;

extracting the root, . . . $8y^2 - 245 = \pm 213$;

$$\therefore y^2 = 57\tfrac{1}{4}, \text{ or } 4;$$

consequently, $y = \sqrt{57\frac{1}{4}}$, or 2.

Hence, $x - 2y = 3$, $x - y = 5$, $x = 7$, $x + y = 9$, and $x + 2y = 11$.

Ex. 14. Let the numbers be $\frac{x}{y}$, x, and xy.

Then, $\frac{x}{y} + x + xy = 13$,

and, $(\frac{x}{y} + xy)\,x = 30$;

from 1st Eq. . . . . . $\frac{x}{y} + xy = 13 - x$, . . . (A)

from 2d Eq. . . . . . $\frac{x}{y} + xy = \frac{30}{x}$;

consequently, $13 - x = \frac{30}{x}$;

clear. and ch. signs, . . $x^2 - 13x = -30$;

completing (Art. 90), $4x^2 - 52x + (13)^2 = 49$;

extracting the root, . . $2x - 13 = \pm 7$;

$$\therefore x = 10, \text{ or } 3.$$

Substituting this latter value in Equation (A), we have

$$\frac{3}{y} + 3y = 10;$$

clear. and transpo., . . $3y^2 - 10y = -3$;

completing (Art. 90), $36y^2 - 120y + (10)^2 = 64$;

extracting the root, . . $6y - 10 = \pm 8$;

$$\therefore\ y = 3, \text{ or } \tfrac{1}{3}.$$

$$\text{Consequently, } \frac{x}{y} = 1,\ x = 3,\ xy = 9.$$

$$\text{or, } \frac{x}{y} = 9,\ x = 3,\ xy = 1.$$

Ex. 15. Let $x - y$ and $x + y$ be the numbers.

Then, the arithmetical mean will be x (Art. 71. Theor. 2);

the geometrical mean . . . . $\sqrt{x^2 - y^2}$ (Art. 74);

the harmonical mean . . . . $\dfrac{2x^2 - 2y^2}{2x}$ (Art. 79);

and by the question, $\sqrt{x^2 - y^2} = x - 13$ . . . . . . (A)

$$\text{also, } \sqrt{x^2 - y^2} = \frac{2x^2 - 2y^2}{2x} + 12;$$

$$\therefore \frac{2x^2 - 2y^2}{2x} + 12 = x - 13;$$

clearing, . . . $2x^2 - 2y^2 + 24x = 2x^2 - 26x$;

or, transpo. and ch. signs, . . $2y^2 = 50x$;

$$y^2 = 25x.$$

Then, substituting this value of y^2 in Eq. (A), we shall have

$$\sqrt{x^2 - 25x} = x - 13;$$

squaring, . . . $x^2 - 25x = x^2 - 26x + 169$;

or, by transposition, . . . $x = 169$.

$$\text{whence, } (y^2 = 25x),\ y = \sqrt{4225} = \pm 65.$$

Consequently, the two numbers are 104, and 234.

Ex. 16. Let x be the first number, and y the difference of the first and second, then we shall have

$$\begin{aligned} \text{the first} &= x, \\ \text{second} &= x + y, \\ \text{third} &= x + 2y + 5. \end{aligned}$$

But, by question, . . . $3x + 3y + 5 = 20$;

$$\therefore\ x = \frac{15 - 3y}{3} = 5 - y.$$

Also, $x(x + y)(x + 2y + 5) = 130$;

or, by substituting the preceding value of x in this equation, we have

$$(5 - y)(5)(10 + y) = 130;$$

$\div$ by 5 and ch. signs, $y^2 + 5y - 50 = -26$;

transpo. and compl., . . $y^2 + 5y + \frac{25}{4} = \frac{121}{4}$;

extracting the root, . . $y + \frac{5}{2} = \pm \frac{11}{2}$;

$$\therefore\ y = 3.$$

Consequently, $x\ (= 5 - y) = 2$:

and the numbers are 2, 5, and 13.

Ex. 17. Let the circumference of the hind-wheel be x yards, and that of the fore-wheel y yards.

Then, $\frac{120}{x}$ = the number of revolutions of the hind-wheel,

and $\frac{120}{y}$ = ... of the fore-wheel;

but, by question, . . $\frac{120}{y} = \frac{120}{x} + 6$;

or, $\div$ by 6, ... $\dfrac{20}{y} = \dfrac{20}{x} + 1$:

clearing, ... $20x = 20y + xy$;

$$\text{or, } (20 - y)\,x = 20y;$$

$$\therefore\ x = \frac{20y}{20 - y}.$$

But if the circumference of the hind-wheel be $x + 1$, and that of the fore-wheel $y + 1$, then

$$\frac{120}{x+1} = \text{the number of revolutions of the hind-wheel,}$$

$$\text{and } \frac{120}{y+1} = \ldots\ldots\ldots \text{ of the fore wheel.}$$

then, by the question, . $\dfrac{120}{y+1} = \dfrac{120}{x+1} + 4$;

or $\div$ by 4, ... $\dfrac{30}{y+1} = \dfrac{30}{x+1} + 1$;

clearing, . . $30x + 30 = 30y + 30 + xy + x + y + 1$;

or, by transposition, . $(29 - y)\,x = 31y + 1$;

$$\therefore\ x = \frac{31y + 1}{29 - y}.$$

$$\text{Consequently, } \frac{31y + 1}{29 - y} = \frac{20y}{20 - y};$$

clearing, . . $620y - 31y^2 + 20 - y = 580y - 20y^2$;

transpo. and ch. signs, . . $11y^2 - 39y = 20$;

$\div$ by 11, and compl., $y^2 - \dfrac{39}{11}y + \left(\dfrac{39}{22}\right)^2 = \dfrac{20}{11} + \dfrac{1521}{484} = \dfrac{2401}{(22)^2}$;

extracting the root, . . . $y-\frac{39}{22}=\pm\frac{49}{22}$;

$$\therefore y=4, \text{ or } -\frac{5}{11}:$$

whence $x\left(=\frac{20y}{20-y}\right)=5.$

REDUCTION OF SURDS.

PROBLEM I. *To reduce a Rational Quantity to the Form of a Surd.*

Page 147.

Ex. 3. Here, $(a^2x^3)^5=a^{10}x^{15}$; $\therefore a^2x^3=\sqrt[5]{a^{10}x^{15}}$.

Ex. 4. Here, $\left(\frac{x^2}{y^3}\right)^4=\frac{x^8}{y^{12}}$, $\therefore \frac{x^2}{y^3}=\left(\frac{x^8}{y^{12}}\right)^{\frac{1}{4}}$.

Ex. 5. $\left(\frac{\sqrt{a}}{x}\right)^3=\frac{a\sqrt{a}}{x^3}$, $\therefore \frac{\sqrt{a}}{x}=\left(\frac{a\sqrt{a}}{x^3}\right)^{\frac{1}{3}}$;

*or otherwise, $\left(\frac{a^{\frac{1}{2}}}{x}\right)^3=\frac{a^{\frac{3}{2}}}{x^3}$, $\therefore \frac{a^{\frac{1}{2}}}{x}=\left(\frac{a^{\frac{3}{2}}}{x^3}\right)^{\frac{1}{3}}$.

Ex. 6. $(a^{\frac{1}{3}}x^{\frac{1}{4}})^2=a^{\frac{2}{3}}x^{\frac{1}{2}}$, $\therefore a^{\frac{1}{3}}x^{\frac{1}{4}}=(a^{\frac{2}{3}}x^{\frac{1}{2}})^{\frac{1}{2}}$.

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* See Note in the Algebra, page 7.

PROBLEM II. *To reduce Surds expressing different Roots to equivalent ones expressing the same Root.*

Page 148.

Ex. 3. Here the indices, brought to a common denominator, are

$\frac{4}{12}$ and $\frac{3}{12}$; hence, $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}} = 4^{\frac{4}{12}}$ and $5^{\frac{3}{12}} = \sqrt[12]{256}$ and $\sqrt[12]{125}$.

Ex. 4. $2\sqrt[3]{3}$ and $3\sqrt{2}$, or $2\,(3)^{\frac{1}{3}}$ and $3\,(2)^{\frac{1}{2}} = 2 \times 3^{\frac{2}{6}}$ and $3 \times 2^{\frac{3}{6}}$

$= 2\sqrt[6]{9}$ and $3\sqrt[6]{8}$.

Ex. 5. $6^{\frac{2}{3}}$ and $5^{\frac{3}{4}} = 6^{\frac{8}{12}}$ and $5^{\frac{9}{12}} = (1679616)^{\frac{1}{12}}$ and $(1953125)^{\frac{1}{12}}$.

Ex. 6. $x^{\frac{3}{2}}$ and $y^{\frac{1}{5}} = x^{\frac{15}{10}}$ and $y^{\frac{2}{10}} = (x^{15})^{\frac{1}{10}}$ and $(y^2)^{\frac{1}{10}}$.

PROBLEM III. *To reduce Surds to their most Simple Forms.*

Page 149.

Ex. 4. $3\sqrt[3]{108} = 3\sqrt[3]{27 \times 4} = 3 \times 3\sqrt[3]{4} = 9\sqrt[3]{4}$.

Ex. 5. $\sqrt[3]{ax^3 + bx^6} = \sqrt[3]{x^3\,(a + bx^3)} = x\sqrt[3]{a + bx^3}$, or $x\,(a + bx^3)^{\frac{1}{3}}$.

Ex. 6. $\sqrt[3]{5\,(a^3 + a^4b)} = \sqrt[3]{5a^3\,(1 + ab)} = a\sqrt[3]{5\,(1 + ab)}$,

or $a\,\{5\,(1 + ab)\}^{\frac{1}{3}}$.

Surds of fractional Form.

Ex. 3. $\frac{a}{b}\sqrt{\frac{c^2}{d}} = \frac{a}{b}\sqrt{\frac{c^2d}{d^2}} = \frac{ac}{bd}\sqrt{d}$.

Ex. 4. $5\sqrt[3]{\frac{2}{3}} = 5\sqrt[3]{\frac{18}{27}} = 5\sqrt[3]{\frac{1}{27}\times 18} = \frac{5}{3}\sqrt[3]{18}.$

Ex. 5. $\frac{1}{3}\sqrt[3]{\frac{5}{9}} = \frac{1}{3}\sqrt[3]{\frac{15}{27}} = \frac{1}{3}\,\frac{\sqrt[3]{15}}{3} = \frac{1}{9}\sqrt[3]{15}.$

Ex. 6. $\sqrt{\frac{ab^2}{4(a+x)}} = \sqrt{\frac{ab^2(a+x)}{4(a+x)^2}} = \frac{\sqrt{ab^2(a+x)}}{2(a+x)} =$

$$\frac{b}{2(a+x)}\sqrt{a(a+x)}.$$

ADDITION AND SUBTRACTION OF SURDS.

Page 150.

Ex. 4. $$\left.\begin{aligned} &\sqrt[3]{192} = \sqrt[3]{64\times 3} = 4\sqrt[3]{3} \\ \text{and } &\sqrt[3]{24} = \sqrt[3]{8\times 3} = 2\sqrt[3]{3} \end{aligned}\right\} \therefore 2\sqrt[3]{3} = \text{difference.}$$

Ex. 5. $$\left.\begin{aligned} &3\sqrt{\frac{2}{5}} = 3\sqrt{\frac{10}{25}} = 3\,\frac{\sqrt{10}}{5} = \frac{3}{5}\sqrt{10} \\ \text{and } &2\sqrt{\frac{1}{10}} = 2\sqrt{\frac{10}{100}} = 2\,\frac{\sqrt{10}}{10} = \frac{1}{5}\sqrt{10} \end{aligned}\right\} \therefore \frac{4}{5}\sqrt{10} = \text{sum.}$$

Ex. 6. $$\left.\begin{aligned} &\sqrt{\frac{8}{27}} = \sqrt{\frac{24}{81}} = \frac{\sqrt{4\times 6}}{9} = \frac{2}{9}\sqrt{6} \\ \text{and } &\sqrt{\frac{1}{6}} = \sqrt{\frac{6}{36}} = \frac{\sqrt{6}}{6} = \frac{1}{6}\sqrt{6} \end{aligned}\right\} \therefore \frac{1}{18} = \text{diff.}$$

Ex. 7 $$\left.\begin{aligned} &\sqrt{24} = \sqrt{4\times 6} = 2\sqrt{6} \\ &2\sqrt{72} = 2\sqrt{36\times 2} = 2\times 6\sqrt{2} = 12\sqrt{2} \\ \text{and } &a\sqrt{bx^2} = \ldots\ldots\ldots\ldots\ ax\sqrt{b} \end{aligned}\right\} = \text{sum.}$$

Ex. 8. $$\left.\begin{aligned} &\sqrt[3]{500} = \sqrt[3]{125 \times 4} = 5\sqrt[3]{4} \\ \text{and } &\sqrt[3]{108} = \sqrt[3]{27 \times 4} = 3\sqrt[3]{4} \end{aligned}\right\} \therefore 8\sqrt[3]{4} = \text{sum.}$$

Ex. 9. $$\left.\begin{aligned} &3\sqrt{\frac{2}{5}} = 3\sqrt{\frac{10}{25}} = 3\,\frac{\sqrt{10}}{5} = \frac{3}{5}\sqrt{10} \\ \text{and } &2\sqrt{\frac{1}{10}} = 2\sqrt{\frac{10}{100}} = 2\,\frac{\sqrt{10}}{10} = \frac{1}{5}\sqrt{10} \end{aligned}\right\} \therefore \frac{2}{5}\sqrt{10} = \text{diff.}$$

Ex. 10. $$\left.\begin{aligned} &\frac{3}{4}\sqrt{\frac{2}{3}} = \frac{3}{4}\sqrt{\frac{6}{9}} = \frac{3}{4}\,\frac{\sqrt{6}}{3} = \frac{3}{12}\sqrt{6} = \frac{1}{4}\sqrt{6} \\ \text{and } &\frac{2}{5}\sqrt{\frac{1}{6}} = \frac{2}{5}\sqrt{\frac{6}{36}} = \frac{2}{5}\,\frac{\sqrt{6}}{6} = \frac{2}{30}\sqrt{6} = \frac{1}{15}\sqrt{6} \end{aligned}\right\}$$

$$\therefore \frac{11}{60}\sqrt{6} = \text{diff.}$$

Ex. 11. $$\left.\begin{aligned} &5\sqrt{20} = 5\sqrt{4 \times 5} = 10\sqrt{5} \\ \text{and } &3\sqrt{45} = 3\sqrt{9 \times 5} = 9\sqrt{5} \end{aligned}\right\} \therefore \sqrt{5} = \text{diff.}$$

Ex. 12. $$\left.\begin{aligned} &\sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3} \\ &\sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3} \\ &4\sqrt{147} = 4\sqrt{49 \times 3} = 28\sqrt{3} \\ \text{and } &3\sqrt{75} = 3\sqrt{25 \times 3} = 15\sqrt{3} \end{aligned}\right\} \therefore 50\sqrt{3} = \text{sum.}$$

MULTIPLICATION AND DIVISION OF SURDS.

Page 151.

Ex. 3. $$2\sqrt[3]{\frac{2}{3}} \times 3\sqrt[3]{\frac{5}{6}} = 6\sqrt[3]{\frac{2}{3} \times \frac{5}{6}} = 6\sqrt[3]{\frac{5}{9}} = 6\sqrt[3]{\frac{15}{27}}$$

$$= 6\,\frac{\sqrt[3]{15}}{3} = \frac{6}{3}\sqrt[3]{15} = 2\sqrt[3]{15}.$$

Ex. 4. $\dfrac{4(ax)^{\frac{1}{3}}}{3(xy)^{\frac{1}{2}}} = \dfrac{4(ax)^{\frac{2}{6}}}{3(xy)^{\frac{3}{6}}} = \dfrac{4}{3}\sqrt[6]{\dfrac{(ax)^2}{(xy)^3}} = \dfrac{4}{3}\sqrt[6]{\dfrac{a^2x^2}{x^3y^3}} = \dfrac{4}{3}\sqrt[6]{\dfrac{a^2}{xy^3}}$.

Ex. 5. $4(3)^{\frac{1}{2}} \times 3(4)^{\frac{1}{3}} = 4(3)^{\frac{3}{6}} \times 3(4)^{\frac{2}{6}} = 12\sqrt[6]{(3)^3 \times (4)^2}$

$$= 12\sqrt[6]{432}.$$

Ex. 6. $\dfrac{4(32)^{\frac{1}{6}}}{(16)^{\frac{1}{3}}} = 4\,\dfrac{(32)^{\frac{1}{6}}}{(16)^{\frac{2}{6}}} = 4\left(\dfrac{32}{256}\right)^{\frac{1}{6}} = 4\left(\dfrac{32}{2^6 \times 4}\right)^{\frac{1}{6}} = \dfrac{4}{2}\left(\dfrac{32}{4}\right)^{\frac{1}{6}}$

$$= 2(8)^{\frac{1}{6}} = 2(2^3)^{\frac{1}{6}} = 2(2)^{\frac{3}{6}} = 2(2)^{\frac{1}{2}} \text{ or } 2\sqrt{2}.$$

Ex. 7. $5a^{\frac{1}{2}} \times 3a^{\frac{1}{3}} = 5a^{\frac{3}{6}} \times 3a^{\frac{2}{6}} = 15a^{\frac{5}{6}}$ or $15\sqrt[6]{a^5}$.

Ex. 8. $2\sqrt{27} \times \sqrt{3} = 2\sqrt{27 \times 3} = 18$.

Ex. 9. $\dfrac{\frac{1}{2}\sqrt{5}}{\frac{2}{3}\sqrt{2}} = \dfrac{3}{4}\sqrt{\dfrac{10}{4}} = \dfrac{3}{8}\sqrt{10}$.

To extract the Square Root of a Binomial Surd.

Page 153.

Ex. 3. Here, $a = 6$, and $b = 20$;

$$\therefore \sqrt{\left\{\frac{a + \sqrt{(a^2 - b)}}{2}\right\}} = \sqrt{\left\{\frac{6 + \sqrt{(36 - 20)}}{2}\right\}} = \sqrt{5};$$

$$\text{and } \sqrt{\left\{\frac{a - \sqrt{(a^2 - b)}}{2}\right\}} = \sqrt{\left\{\frac{6 - \sqrt{(36 - 20)}}{2}\right\}} = \sqrt{1} = 1;$$

$$\therefore \sqrt{(6 + \sqrt{20})} = 1 + \sqrt{5}.$$

Ex. 4. Here, $6 - 2\sqrt{5} = 6 - \sqrt{4 \times 5} = 6 - \sqrt{20}$; hence $a = 6$, and $b = 20$;

$$\therefore \sqrt{\left\{\frac{6+\sqrt{(36-20)}}{2}\right\}} = \sqrt{5},$$

$$\text{and } \sqrt{\left\{\frac{6-\sqrt{(36-20)}}{2}\right\}} = \sqrt{1} = 1;$$

Consequently, $\sqrt{(6-2\sqrt{5})} = \sqrt{5} - 1$.

Ex. 5. Here, $7 - 2\sqrt{10} = 7 - \sqrt{4 \times 10} = 7 - \sqrt{40}$; hence $a = 7$, and $b = 40$;

$$\therefore \sqrt{\left\{\frac{7+\sqrt{(49-40)}}{2}\right\}} = \sqrt{5};$$

$$\text{and } \sqrt{\left\{\frac{7-\sqrt{(49-40)}}{2}\right\}} = \sqrt{2};$$

$$\therefore \sqrt{(7-2\sqrt{10})} = \sqrt{5} - \sqrt{2}.$$

Ex. 6. Here, $42 + 3\sqrt{174\frac{2}{9}} = 42 + \sqrt{9 \times 174\frac{2}{9}} = 42 + \sqrt{1568}$; whence $a = 42$, and $b = 1568$;

$$\therefore \sqrt{\left\{\frac{42+\sqrt{(1764-1568)}}{2}\right\}} = \sqrt{28};$$

$$\text{and } \sqrt{\left\{\frac{42-\sqrt{(1764-1568)}}{2}\right\}} = \sqrt{14}.$$

Consequently, $\sqrt{(42 + 3\sqrt{174\frac{2}{9}})} = \sqrt{28} + \sqrt{14}$.

To find Multipliers which will make Binomial Surds rational.

Page 156.

Ex. 3. Here, $a = 5$, $b = 2$, and $n = 3$; consequently, the multiplier $\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}}\,b + \&c. = \sqrt[3]{25} + \sqrt[3]{10} + \sqrt[3]{4}$, and by multiplying,

$$\begin{array}{lllll}
 & \sqrt[3]{5} - \sqrt[3]{2} & & & \\
\hline
 & \sqrt[3]{125} & + \sqrt[3]{50} & + \sqrt[3]{20} & \\
 & & - \sqrt[3]{50} & - \sqrt[3]{20} & - \sqrt[3]{8} \\
\hline
\text{we obtain} & \sqrt[3]{125} & \quad\cdot & \quad\cdot & - \sqrt[3]{8} = 5 - 2; \\
\hline
\end{array}$$

hence $\dfrac{3}{\sqrt[3]{5}-\sqrt[3]{2}}=\dfrac{3(\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4})}{(\sqrt[3]{5}-\sqrt[3]{2})(\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4})}$

$$=\frac{3(\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4})}{5-2}=\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4}.$$

Ex. 4. Here, $n=2$; $\therefore$ the multiplier $\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+$ &c. $=\sqrt{a}-\sqrt{b}$; then multiplying the numerator and denominator by $\sqrt{a}-\sqrt{b}$, we have

$$\frac{a}{\sqrt{a}+\sqrt{b}}=\frac{a(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})}=\frac{a(\sqrt{a}-\sqrt{b})}{a-b}.$$

Ex. 5. Here $n=3$; $\therefore$ the multiplier becomes

$$\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2};$$

and, multiplying the numerator and denominator by this, will give

$$\frac{a}{\sqrt[3]{x}+\sqrt[3]{y}}=\frac{a(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2})}{(\sqrt[3]{x}+\sqrt[3]{y})(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2})}$$

$$=\frac{a(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2})}{x+y}.$$

Ex. 6. Here, $a=3$, $b=4$, and $n=4$, consequently, the multiplier $\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-$ &c.

$=\sqrt[4]{27}-\sqrt[4]{36}+\sqrt[4]{48}-\sqrt[4]{64}$; and by multiplying,

$$\begin{array}{l}
\sqrt[4]{3}+\sqrt[4]{4} \\
\hline
\sqrt[4]{81}-\sqrt[4]{108}+\sqrt[4]{144}-\sqrt[4]{192} \\
\qquad\qquad \sqrt[4]{108}-\sqrt[4]{144}+\sqrt[4]{192}-\sqrt[4]{256} \\
\hline
\end{array}$$

we obtain $\sqrt[4]{81}\quad\cdot\quad\cdot\quad\cdot\quad-\sqrt[4]{256}=3-4.$

ON IMAGINARY QUANTITIES.

MULTIPLICATION.

Page 160.

Ex. 3. Here $4\sqrt{-5} \times 3\sqrt{-1} = -12\sqrt{5}$.

Ex. 4. $-5\sqrt{-2} \times -3\sqrt{-5} = -15\sqrt{10}$.

Ex. 5. $(4 + \sqrt{-3}) \times \sqrt{-5} = 4\sqrt{-5} - \sqrt{15}$.

Ex. 6.

$$\begin{array}{l}
a - b\sqrt{-1} \\
a - b\sqrt{-1} \\
\hline
a^2 - ab\sqrt{-1} \\
\quad - ab\sqrt{-1} - b^2 \\
\hline
a^2 - 2ab\sqrt{-1} - b^2 \\
a - b\sqrt{-1} \\
\hline
a^3 - 2a^2b\sqrt{-1} - ab^2 \\
\quad - a^2b\sqrt{-1} - 2ab^2 + b^3\sqrt{-1} \\
\hline
\end{array}$$

$\therefore (a - b\sqrt{-1})^3 = a^3 + b^3\sqrt{-1} - 3ab(b + a\sqrt{-1})$.

DIVISION.

Page 161.

Ex. 3. $\dfrac{2\sqrt{-7}}{-3\sqrt{-5}} = -\dfrac{2}{3}\sqrt{\dfrac{7}{5}}$.

Ex. 4. $\dfrac{-\sqrt{-1}}{-6\sqrt{-3}} = \dfrac{1}{6}\sqrt{\dfrac{1}{3}}$, or $\dfrac{1}{6\sqrt{3}}$.

Ex. 5. $\dfrac{4+\sqrt{-2}}{2-\sqrt{-2}} = \dfrac{(4+\sqrt{-2})\,(2+\sqrt{-2})}{(2-\sqrt{-2})\,(2+\sqrt{-2})} = \dfrac{6+6\sqrt{-2}}{6}$

$= 1+\sqrt{-2}$.

Ex. 6. $\dfrac{3+2\sqrt{-1}}{3-2\sqrt{-1}} = \dfrac{(3+2\sqrt{-1})^2}{(3-2\sqrt{-1})\,(3+2\sqrt{-1})} =$

$\dfrac{5+12\sqrt{-1}}{13} = \dfrac{1}{13}(5+12\sqrt{-1})$.

ON THE BINOMIAL THEOREM.

APPLICATION OF THE BINOMIAL THEOREM TO THE EXPANSION OF SERIES.

1. *To expand* $(a+x)^m$ *when* m *is a Positive, or Negative Integer.*

Page 173.

Ex. 5. Here, the first term is . . . . x^7.

Second . . . . $-7x^6y$.

Third . . . . $\dfrac{7\times 6}{2}x^5y^2 = 21x^5y^2$.

Fourth . . . . $-\dfrac{21\times 5}{3}x^4y^3 = -35x^4y^3.$

Fifth . . . . $\dfrac{35\times 4}{4}x^3y^4 = \quad 35x^3y^4.$

Sixth . . . . $-\dfrac{35\times 3}{5}x^2y^5 = -21x^2y^5.$

Seventh . . . . $\dfrac{21\times 2}{6}xy^6 = \quad 7xy^6.$

Eighth . . . . $-\dfrac{7\times 1}{7}y^7 = -y^7.$

Hence $(x-y)^7 = x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5$
$+ 7xy^6 - y^7.$

Ex 6. Here the first term is . . . . $x^7.$

the Second . . . . $7x^6\times(2y) = 14x^6y.$

Third . . . . $\dfrac{7\times 6}{2}x^5(2y)^2 = 84x^5y^2.$

Fourth . . . . $\dfrac{21\times 5}{3}x^4(2y)^3 = 280x^4y^3.$

Fifth . . . . $\dfrac{35\times 4}{4}x^3(2y)^4 = 560x^3y^4.$

Sixth . . . . $\dfrac{35\times 3}{5}x^2(2y)^5 = 672x^2y^5.$

Seventh . . . . $\dfrac{21\times 2}{6}x\,(2y)^6 = 448\,xy^6.$

Eighth . . . . $\dfrac{7\times 1}{7}(2y)^7 = 128y^7.$

$$\therefore (x+2y)^7 = \begin{cases} x^7 + 14x^6y + 84x^5y^2 + 280x^4y^3 + 560x^3y^4 + 672x^2y^5 + \\ 448xy^6 + 128y^7. \end{cases}$$

Ex. 7. Here, the first term is $(a+b)^3$, or $a^3 + 3a^2b + 3ab^2 + b^3.$

the Second . . . . $3(a+b)^2c$, or $3a^2c + 6abc + 3b^2c.$

Third . . . $\frac{3 \times 2}{2}(a+b)\,c^2 = 3\,(a+b)\,c^2$, or $3ac^2 + 3bc^2$.

Fourth . . . . $\frac{3 \times 1}{3}\,c^3 = c^3$.

Whence $\{(a+b)+c\}^3$, or $(a+b+c)^3 =$

$(a+b)^3 + 3\,(a+b)^2 c + 3\,(a+b)\,c^2 + c^3$, or

$a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

Ex. 8. Here $\frac{2}{(c+x)^2} = \frac{1}{(c+x)^2} \times 2 = 2\,(c+x)^{-2}$; and expanding $(c+x)^{-2}$, we have, for the first term, ... $c^{-2} = \frac{1}{c^2}$.

For the second . . . . $-2c^{-3}x = -\frac{2x}{c^3}$.

Third . . . . $\frac{2 \times 3}{2}\,c^{-4}x^2 = 3c^{-4}x^2 = \frac{3x^2}{c^4}$.

Fourth . . . . $-\frac{3 \times 4}{3}\,c^{-5}x^3 = -4c^{-5}x^3 = -\frac{4x^3}{c^5}$.

&c. &c. &c.

Hence $\frac{1}{(c+x)^2} = \frac{1}{c^2} - \frac{2x}{c^3} + \frac{3x^2}{c^4} - \frac{4x^3}{c^5} + $ &c.

$\therefore \frac{2}{(c+x)^2} = \frac{2}{c^2} - \frac{4x}{c^3} + \frac{6x^2}{c^4} - \frac{8x^3}{c^5} + $ &c.

Ex. 9. Here $\frac{a^2}{(a+2b)^3} = a^2 \times \frac{1}{(a+2b)^3} = a^2\,(a+2b)^{-3}$; and by expansion, we have, for the first term, ... $a^{-3} = \frac{1}{a^3}$.

For the second . . . . $-3a^{-4}\,(2b) = -6a^{-4}b = -\frac{6b}{a^4}$.

Third . . . . . . $\frac{3 \times 4}{2}\,a^{-5}\,(2b)^2 = 24a^{-5}\,b^2 = \frac{24b^2}{a^5}$.

Fourth . . . $-\dfrac{6\times 5}{3}a^{-6}(2b)^3 = -80a^{-6}b^3 = -\dfrac{80b^3}{a^6}$.

&c. &c. &c.;

hence $\dfrac{1}{(a+2b)^3} = \dfrac{1}{a^3} - \dfrac{6b}{a^4} + \dfrac{24b^2}{a^5} - \dfrac{80b^3}{a^6} + \text{\&c.}$

$$\therefore \frac{a^2}{(a+2b)^3} = \frac{a^2}{a^3} - \frac{6a^2b}{a^4} + \frac{24a^2b^2}{a^5} - \frac{80a^2b^3}{a^6} + \text{\&c.}$$

$$= \frac{1}{a}\left(1 - \frac{6b}{a} + \frac{24b^2}{a^2} - \frac{80b^3}{a^3} + \text{\&c}\right).$$

2. *To expand* $(a+x)^{\frac{m}{n}}$, $\frac{m}{n}$ *being either Positive or Negative.*

Page 178.

Ex. 6. Here $(a+x)^{\frac{m}{n}} = (b^2+x)^{\frac{1}{2}}$, $\therefore a = b^2$, $m = 1$, $n = 2$, and

$$Q = \frac{x}{b^2}.$$

Whence $a^{\frac{m}{n}} = (b^2)^{\frac{1}{2}} = b = A$.

$$\frac{m}{n}AQ = \frac{1}{2}b \times \frac{x}{b^2} = \frac{x}{2b} = B.$$

$$\frac{m-n}{2n}BQ = -\frac{1}{4} \times \frac{x}{2b} \times \frac{x}{b^2} = -\frac{x^2}{2.4b^3} = C.$$

$$\frac{m-2n}{3n}CQ = -\frac{3}{6} \times -\frac{x^2}{2.4b^3} \times \frac{x}{b^2} = \frac{3x^3}{2.4.6b^5} = D.$$

$$\frac{m-3n}{4n}DQ = -\frac{5}{8} \times \frac{3x^3}{2.4\,6b^5} \times \frac{x}{b^2} = -\frac{3.5x^4}{2.4.6.8b^7} = E.$$

&c. &c. &c.

$$\therefore \sqrt{b^2+x} = b + \frac{x}{2b} - \frac{x^2}{2.4b^3} + \frac{3x^3}{2.4.6b^5} - \frac{3.5x^4}{2.4.6.8b^7} + \text{\&c.}$$

By Rule.

The general development of $(a-x)^m$ is

$$(a-x)^m = a^m - ma^{m-1}x + \frac{m(m-1)}{2}a^{m-2}x^2 +$$

$$\frac{m(m-1)(m-2)}{2.3}a^{m-3}x^3 + \&c.,$$

which, for the particular exponent $\frac{1}{2}$, is

$$(a-x)^{\frac{1}{2}} = a^{\frac{1}{2}} - \tfrac{1}{2}a^{-\frac{1}{2}}x + \frac{\frac{1}{2}\times-\frac{1}{2}}{2}a^{-\frac{3}{2}}x^2 + \frac{\frac{1}{2}\times-\frac{1}{2}\times-\frac{3}{2}}{2.3}a^{-\frac{5}{2}}x^3 + \&c.,$$

in which, if b^2 be put for a, we shall have

$$(b^2-x)^{\frac{1}{2}} = b - \tfrac{1}{2}b^{-1}x - \frac{1}{2.4}b^{-3}x^2 - \frac{3}{2.4.6}b^{-5}x^3 - \&c.$$

And in this manner may all the following examples be solved, without reference to the preceding formula (Art. 134).

Ex. 7. Here $\dfrac{c^2}{(c^2-x)^{\frac{1}{2}}} = \dfrac{1}{(c^2-x)^{\frac{1}{2}}} \times c^2 = c^2(c^2-x)^{-\frac{1}{2}}$;

$$\therefore P = c^2,\ m = -1,\ n = 2, \text{ and } Q = -\frac{x}{c^2}.$$

Whence $P^{\frac{m}{n}} = (c^2)^{-\frac{1}{2}} = \dfrac{1}{(c^2)^{\frac{1}{2}}} = \dfrac{1}{c} = A.$

$$\frac{m}{n}AQ = -\frac{1}{2}\times\frac{1}{c}\times-\frac{x}{c^2} = \frac{x}{2c^3} = B.$$

$$\frac{m-n}{2n}BQ = -\frac{3}{4}\times\frac{x}{2c^3}\times-\frac{x}{c^2} = \frac{3x^2}{2.4c^5} = C.$$

$$\frac{m-2n}{3n}CQ = -\frac{5}{6}\times\frac{3x^2}{2.4c^5}\times-\frac{x}{c^2} = \frac{3.5x^3}{2.4.6c^7} = D.$$

$$\frac{m-3n}{4n}DQ = -\frac{7}{8}\times\frac{3.5x^3}{2.4.6c^7}\times-\frac{x}{c^2} = \frac{3.5.7x^4}{2.4.6.8c^9} = E.$$

&c. &c. &c.

Hence $\dfrac{1}{(c^2-x)^{\frac{1}{2}}} = \dfrac{1}{c} + \dfrac{x}{2c^3} + \dfrac{3x^2}{2.4c^5} + \dfrac{3.5x^3}{2.4.6c^7} + \dfrac{3.5.7x^4}{2.4.6.8c^9} + \text{\&c.};$

$$\therefore \frac{c^2}{(c^2-x)^{\frac{1}{2}}} = c^2\left(\frac{1}{c} + \frac{x}{2c^3} + \frac{3x^2}{2.4c^5} + \frac{3.5x^3}{2.4.6c^7} + \frac{3.5.7x^4}{2.4.6.8c^9} + \text{\&c.}\right)$$

$$= c + \frac{x}{2c} + \frac{3x^2}{2.4c^3} + \frac{3.5x^3}{2.4.6c^5} + \frac{3.5.7x^4}{2.4.6.8c^7} + \text{\&c.}$$

Ex. 8. Here $m = 2$, $n = 3$, and $Q = \dfrac{x}{a}$.

Whence $a^{\frac{m}{n}} = a^{\frac{2}{3}} = A$.

$$\frac{m}{n} AQ = \frac{2}{3} \times a^{\frac{2}{3}} \times \frac{x}{a} = \frac{2a^{\frac{2}{3}}x}{3a} = B.$$

$$\frac{m-n}{2n} BQ = -\frac{1}{6} \times \frac{2a^{\frac{2}{3}}x}{3a} \times \frac{x}{a} = -\frac{2a^{\frac{2}{3}}x^2}{3.6a^2} = C.$$

$$\frac{m-2n}{3n} CQ = -\frac{4}{9} \times -\frac{2a^{\frac{2}{3}}x^2}{3.6a^2} \times \frac{x}{a} = \frac{2.4a^{\frac{2}{3}}x^3}{3.6.9a^3} = D.$$

$$\frac{m-3n}{4n} DQ = -\frac{7}{12} \times \frac{2.4a^{\frac{2}{3}}x^3}{3.6.9a^3} \times \frac{x}{a} = -\frac{2.4.7a^{\frac{2}{3}}x^4}{3.6.9.12a^4} = E.$$

&c. &c. &c.

$$\therefore (a+x)^{\frac{2}{3}} = a^{\frac{2}{3}} + \frac{2a^{\frac{2}{3}}x}{3a} - \frac{2a^{\frac{2}{3}}x^2}{3.6a^2} + \frac{2.4a^{\frac{2}{3}}x^3}{3.6.9a^3} - \frac{2.4.7a^{\frac{2}{3}}x^4}{3.6.9.12a^4} + \text{\&c.}$$

$$= a^{\frac{2}{3}}\left(1 + \frac{2x}{3a} - \frac{x^2}{3^2a^2} + \frac{4x^3}{3^4a^3} - \frac{7x^4}{3^5a^4} + \text{\&c.}\right)$$

Ex. 9. Here $\sqrt[3]{9} = (8+1)^{\frac{1}{3}}$ $\therefore a = 8$, $x = 1$, $m = 1$, $n = 3$, and

$$Q = \frac{1}{8} = \frac{1}{2^3}.$$

M 2

Whence $a^{\frac{m}{n}} = 8^{\frac{1}{3}} = 2 = A.$

$$\frac{m}{n} AQ = \frac{1}{3} \times 2 \times \frac{1}{2^3} = \frac{2}{3.2^3} = \frac{1}{3.2^2} = B.$$

$$\frac{m-n}{2n} BQ = -\frac{2}{6} \times \frac{1}{3.2^2} \times \frac{1}{2^3} = -\frac{1}{3.6.2^4} = C.$$

$$\frac{m-2n}{3n} CQ = -\frac{5}{9} \times -\frac{1}{3.6.2^4} \times \frac{1}{2^3} = \frac{5}{3.6.9.2^7} = D.$$

$$\frac{m-3n}{4n} DQ = -\frac{8}{12} \times \frac{5}{3.6.9.2^7} \times \frac{1}{2^3} = -\frac{5.8}{3.6.9.12.2^{10}} = E.$$

&c. &c. &c.

Hence $\sqrt[3]{9} = 2 + \frac{1}{3.2^2} - \frac{1}{3.6.2^4} + \frac{5}{3.6.9.2^7} - \frac{5.8}{3.6.9.12.2^{10}} + \&c.$

Ex. 10. Here $\sqrt{2} = (1+1)^{\frac{1}{2}} \therefore a = 1,\ x = 1,\ m = 1,\ n = 2$, and

$$Q = \frac{1}{1} = 1.$$

Whence $a^{\frac{m}{n}} = 1^{\frac{1}{2}} = 1 = A.$

$$\frac{m}{n} AQ = \frac{1}{2} \times 1 \times 1 = \frac{1}{2} = B.$$

$$\frac{m-n}{2n} BQ = -\frac{1}{4} \times \frac{1}{2} = -\frac{1}{2.4} = C.$$

$$\frac{m-2n}{3n} CQ = -\frac{3}{6} \times -\frac{1}{2.4} = \frac{3}{2.4.6} = D.$$

$$\frac{m-3n}{4n} DQ = -\frac{5}{8} \times \frac{3}{2.4.6} = -\frac{3.5}{2.4.6.8} = E.$$

&c. &c. &c.

$$\therefore \sqrt{2} = 1 + \frac{1}{2} - \frac{1}{2.4} + \frac{3}{2.4.6} - \frac{3.5}{2.4.6.8} + \&c.$$

Ex. 11. Here $a = a^2$, $x = -x^2$, $m = 3$, $n = 4$, and $Q = -\frac{x^2}{a^2}$.

Whence $a^{\frac{m}{n}} = (a^2)^{\frac{3}{4}} = a^{\frac{6}{4}} = a^{\frac{3}{2}} = A$.

$$\frac{m}{n}AQ = \frac{3}{4} \times a^{\frac{3}{2}} \times -\frac{x^2}{a^2} = -\frac{3a^{\frac{3}{2}}x^2}{2^2a^2} = B.$$

$$\frac{m-n}{2n}BQ = -\frac{1}{2^3} \times -\frac{3a^{\frac{3}{2}}x^2}{2^2a^2} \times -\frac{x^2}{a^2} = -\frac{3a^{\frac{3}{2}}x^4}{2^5a^4} = C.$$

$$\frac{m-2n}{3n}CQ = -\frac{5}{3.2^2} \times -\frac{3a^{\frac{3}{2}}x^4}{2^5a^4} \times -\frac{x^2}{a^2} = -\frac{5a^{\frac{3}{2}}x^6}{2^7a^6} = D.$$

$$\frac{m-3n}{4n}DQ = -\frac{9}{2^4} \times -\frac{5a^{\frac{3}{2}}x^6}{2^7a^6} \times -\frac{x^2}{a^2} = -\frac{5.9a^{\frac{3}{2}}x^8}{2^{11}a^8} = E.$$

$$\therefore (a^2 - x^2)^{\frac{3}{4}} = a^{\frac{3}{2}} - \frac{3a^{\frac{3}{2}}x^2}{2^2a^2} - \frac{3a^{\frac{3}{2}}x^4}{2^5a^4} - \frac{5a^{\frac{3}{2}}x^6}{2^7a^6} - \frac{5.9a^{\frac{3}{2}}x^8}{2^{11}a^8} - \&c.$$

and, since $\frac{a^{\frac{3}{2}}}{a^2} = \frac{a^{\frac{3}{2}}}{a^{\frac{4}{2}}} = \frac{1}{a^{\frac{1}{2}}}$ or $\frac{1}{\sqrt{a}}$, the preceding series may be written thus:

$$\frac{1}{\sqrt{a}}\left(a^2 - \frac{3x^2}{2^2} - \frac{3x^4}{2^5a^2} - \frac{5x^6}{2^7a^4} - \frac{5.9x^8}{2^{11}a^6} - \&c.\right)$$

APPLICATION OF LOGARITHMS.

Page 193.

Ex. 6. Here $6 \log. 24 + \frac{1}{3} \log. 17 - (\log. 4821 + 4 \log. 6) = 78{\cdot}64561$.

As shown by the following operation:

$1{\cdot}3802112 = \log. 24.$
6
$8{\cdot}2812672$
${\cdot}4101496$
$8{\cdot}6914168$
$6{\cdot}7957423$
$1{\cdot}8956745$
$1{\cdot}8956435 = \log. 78.64$

diff. 552) 3100 (.......561
2760 — $78{\cdot}64561$ Ans.
3400
3312
880
552
328

3) $1{\cdot}2304489 = \log. 17.$
${\cdot}4101496$

${\cdot}7781513 = \log. 6.$
4
$3{\cdot}1126052$
$3{\cdot}6831371 = \log. 4821$
$6{\cdot}7957423$

Ex. 7. Here $\sqrt{\dfrac{284\ \sqrt[3]{621}}{(43)^3}} = \dfrac{\sqrt{284}\ \sqrt[6]{621}}{\sqrt{(43)^3}}$, or $\dfrac{(284)^{\frac{1}{2}}(621)^{\frac{1}{6}}}{(43)^{\frac{3}{2}}}$.

Hence $\frac{1}{2} \log. 284 + \frac{1}{6} \log. 621 - \frac{3}{2} \log. 43$.

2) 2·4533183 = log. 284.

1·2266591
·4655152

1·6921743 = log. of numerator.
2·4502027 = log. of denominator.

—·7580284 = log. of the fraction.

—·7580030 = log. of $\frac{1}{5·728}$.

diff. 759) 2540 (33
2277

2630
2277

353

6) 2·7930916 = log. 621.

·4655152

1·6334685 = log. 43.
3

2) 4·9004055

2·4502027

Hence the answer is $\frac{1}{5·72833}$.

ON EXPONENTIAL EQUATIONS.

Page 196.

Ex. 2. Given $x^x = 5$.

Here x log. x = log. 5 = ·6989700, and upon trial x is found to lie between 2 and 3. Hence, by substituting each of these, we have

2 log. 2 = ·6020600
and 3 log. 3 = 1·4313639

·8293039 = difference of results.

∴ 8293039 : 1 :: ·0969100 : ·0117 nearly.

Whence 2 + ·0117 = 2·0117.

This value we find to be rather too small; and so also is 2·1. Now, by trying 2·2, it is found to be too great; the true value, therefore, is between 2·1 and 2·2.

$$2\cdot1 \text{ log. } 2\cdot1 = \cdot67666053$$
$$2\cdot2 \text{ log. } 2\cdot2 = \cdot75332994$$
$$\cdot07666941 = \text{diff. of results.}$$

$$\therefore \cdot07666941 : \cdot1 :: \cdot0223095 : \cdot029$$

$$\text{and } 2\cdot1 + \cdot029 = 2\cdot129.$$

Ex. 3. Given $x^x = 2000$.

We shall solve this equation by the second method, and shall have

$$\text{log. } 2000 = 3\cdot3010300 = a', \text{ and log. } 3\cdot3010300 = \cdot5186494.$$

$$\therefore x' + \text{log. } x' = \text{log. } a' = \cdot5186494.$$

Now the nearest value of x' in the tables, below the true one, is ·6837, which, added to its log. $\bar{1}\cdot8348656$, gives ·5185656; and the nearest value above the true one is ·6838, which, added to its log. $\bar{1}\cdot8349291$, gives ·5187291.

·5187291	·5186494
·5185656	·5185656
1635	838

$$\therefore 1635 : \cdot0001 :: 838 : 512.$$

$$\text{Consequently, } x' = \cdot6837512 = \text{log. } x;$$

$$\therefore x = 4\cdot8278.$$

COMPOUND INTEREST.

PROBLEM I.

Page 199.

Ex. 7. Here $p = 500$, $a = 900$, $R = 1 + \frac{5}{100} = 1{\cdot}05$;

$$\therefore n = \frac{\log. a - \log. p}{\log. R} = \frac{\log. 900 - \log. 500}{\log. 1{\cdot}05} =$$

$$\frac{2{\cdot}9542425 - 2{\cdot}6989700}{{\cdot}0211893} = 12{\cdot}04 \text{ years.}$$

Ex. 8. Here $p = 200$, $R = 1 + \frac{4}{100} = 1{\cdot}04$, and $n = 7$;

$$\therefore \log. a = \log. p + n \log. R = \log. 200 + 7 \log. 1{\cdot}04.$$

2·3010300 = log. 200.	·0170333 = log. 1·04.
·1192331	7
2·4202631	·1192331
2·4202529 = log. 263·18	

diff. 165) 1020 (. 6
990 — 263·186
30
20
3·72
12
8·64
4
2·56

Hence $a =$ £263. 3*s*. 8½*d*.

Ex. 9. Here $p = £376.\ 17s.\ 9d.$, $a = 1000$, and $n = 20$;

$$\therefore \log. \text{R} = \frac{\log. a - \log. p}{n} = \frac{\log. 1000 - \log. 376{\cdot}887}{20} =$$

$$\frac{3 - 2{\cdot}5762112}{20} = {\cdot}0211894,$$ the number corresponding to which is $1{\cdot}05$;

$$\therefore r = {\cdot}05, \text{ and } {\cdot}05 \times 100 = 5, \text{ the rate per cent.}$$

Ex. 10. Here $m = 2$, and $\text{R} = 1{\cdot}035$;

$$\therefore n = \frac{\log. m}{\log. \text{R}} = \frac{\log. 2}{\log. 1{\cdot}035} = \frac{{\cdot}3010300}{{\cdot}0149403} = 20{\cdot}149 \text{ years.}$$

PROBLEM II.

Page 201.

Ex. 5. Here $\text{A} = 30$, $r = \frac{4{\cdot}5}{100} = {\cdot}045$, and $n = 16$;

$$\therefore a = \frac{\text{A}(\text{R}^n - 1)}{r} = \frac{30\ \{(1{\cdot}045)^{16} - 1\}}{{\cdot}045}.$$

Now, $\log. (1{\cdot}045)^{16} = 16 \log. 1{\cdot}045 = {\cdot}3058608 = \log. 2{\cdot}02237$;

$$\therefore a = \frac{30\ (2{\cdot}02237 - 1)}{{\cdot}045} = \frac{30 \times (1{\cdot}02237)}{{\cdot}045} = £681.\ 11s.\ 7d.$$

Ex. 6. Here $a = \frac{\text{A}(\text{R}^n - 1)}{r}$ $\therefore \frac{ar}{\text{A}} + 1 = \text{R}^n$;

$$\text{and } \log. \left\{ \frac{ar}{\text{A}} + 1 \right\} = n \log. \text{R};$$

$$\therefore n = \log. \left\{ \frac{ar + \text{A}}{\text{A}} \right\} \div \log. \text{R}$$

$$= \log. \left\{ \frac{40 + 20}{20} \right\} \div \log. 1{\cdot}04$$

$$= \text{log. } 3 \div \text{log. } 1{\cdot}04 = \frac{{\cdot}4771213}{{\cdot}0170333} = 28 \text{ years.}$$

Ex. 7. Here $P = \frac{A}{r} = \frac{A}{{\cdot}03} = 33\frac{1}{3}\,A.$

ON SERIES.

THE DIFFERENTIAL METHOD.

PROBLEM I.

To find the first term of any order of differences.

Page 205.

Ex. 3. Here $a, b, c, d,$ &c. $= 1, 8, 27, 64,$ &c. and $n = 3$.

$$\therefore -a + nb - \frac{n(n-1)}{2}c + \frac{n(n-1)(n-2)}{2.3}d =$$

$$-a + 3b - 3c + d = -1 + 24 - 81 + 64 = 6.$$

Ex. 4. Here $a, b, c, d, e,$ &c. $= 1, 6, 20, 50, 105,$ &c. and $n = 4$.

$$\therefore a - nb + \frac{n(n-1)}{2}c - \frac{n(n-1)(n-2)}{2.3}d +$$

$$\frac{n(n-1)(n-2)(n-3)}{2.3.4}e = a - 4b + 6c - 4d + e =$$

$$1 - 24 + 120 - 200 + 105 = 2.$$

PROBLEM II.

To find the *n*th term of the series a, b, c, d, e, &c.

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Ex. 4.

$$\begin{array}{c} 1,\ 3,\ 5,\ 7, \\ 2,\ 2,\ 2, \\ 0,\ 0. \end{array}$$

Here the first terms of the differences are 2, and 0;

that is, $\Delta^1 = 2$, and $\Delta^2 = 0$; also $a = 1$, and $n = 20$;

$$\therefore\ a + (n-1)\,\Delta^1 = 1 + 38 = 39.$$

Ex. 5.

$$\begin{array}{c} 1,\ 3,\ 6,\ 10,\ 15, \\ 2,\ 3,\ 4,\ 5, \\ 1,\ 1,\ 1, \\ 0,\ 0. \end{array}$$

Here $\Delta^1 = 2$, $\Delta^2 = 1$, $\Delta^3 = 0$, $a = 1$, and $n = 20$.

$$\therefore\ a + (n-1)\,\Delta^1 + \frac{(n-1)(n-2)}{2}\,\Delta^2 = 1 + 38 + 171 = 210.$$

Ex. 6.

$$\begin{array}{c} 1,\ 4,\ 9,\ 16,\ 25, \\ 3,\ 5,\ 7,\ 9, \\ 2,\ 2,\ 2, \\ 0,\ 0. \end{array}$$

Here $\Delta^1 = 3$, $\Delta^2 = 2$, $\Delta^3 = 0$, $a = 1$, and $n = 15$;

$$\therefore\ a + (n-1)\,\Delta^1 + \frac{(n-1)(n-2)}{2}\,\Delta^2 = 1 + 42 + 182 = 225.$$

Ex. 7. Here, $f = a - 5b + 10c - 10d + 5e$;

$$\therefore\ d = \frac{a + 10c + 5e - (5b + f)}{10},$$

which is thus calculated:

$$a = \log. 50 = 1{\cdot}6989700$$
$$b = \log. 51 = 1{\cdot}7075702$$
$$c = \log. 52 = 1{\cdot}7160033$$
$$e = \log. 54 = 1{\cdot}7323938$$
$$f = \log. 55 = 1{\cdot}7403627$$

$$\therefore (a + 10c + 5e) = 27{\cdot}5209720$$
$$(5b + f) = 10{\cdot}2782137$$
$$10)\ \overline{17{\cdot}2427583}$$
$$d = \log. 53 = 1{\cdot}7242758$$

PROBLEM III.

To find the sum of n terms of a series.

Page 209.

Ex. 3.

$$1, 2, 3, 4, 5,$$
$$1, 1, 1, 1,$$
$$0, 0, 0.$$

Here $\Delta^1 = 1$, $\Delta^2 = 0$, and $a = 1$;

$$\therefore na + \frac{n(n-1)}{2}\Delta^1 = \frac{n^2+n}{2}.$$

Ex. 4.

$$1, 4, 8, 13, 19,$$
$$3, 4, 5, 6,$$
$$1, 1, 1,$$
$$0, 0.$$

Here, $\Delta^1 = 3$, $\Delta^2 = 1$, $\Delta^3 = 0$, $a = 1$, and $n = 12$;

$$\therefore na + \frac{n(n-1)}{2}\Delta^1 + \frac{n(n-1)(n-2)}{2.3}\Delta^2 =$$

$$12 + 198 + 220 = 430.$$

Ex. 5.

1, 3, 6, 10, 15,

2, 3, 4, 5,

1, 1, 1,

0, 0.

Here $\Delta^1 = 2$, $\Delta^2 = 1$, $\Delta^3 = 0$, and $a = 1$;

$$\therefore\ na + \frac{n(n-1)}{2}\Delta^1 + \frac{n(n-1)(n-2)}{2.3}\Delta^2 =$$

$$n + n^2 - n + \frac{n^3 - 3n^2 + 2n}{6} = \frac{n^3 + 3n^2 + 2n}{6} = \frac{n(n+1)(n+2)}{6}.$$

Ex. 6.

1, 8, 27, 64, 125, 216,

7, 19, 37, 61, 91,

12, 18, 24, 30,

6, 6, 6,

0, 0.

Here $\Delta^1 = 7$, $\Delta^2 = 12$, $\Delta^3 = 6$, $\Delta^4 = 0$, and $a = 1$;

$$\therefore\ na + \frac{n(n-1)}{2}\Delta^1 + \frac{n(n-1)(n-2)}{2.3}\Delta^2 + \frac{n(n-1)(n-2)(n-3)}{2.3.4}\Delta^3$$

$$= n + \frac{7n(n-1)}{2} + 2n(n-1)(n-2) + \frac{n(n-1)(n-2)(n-3)}{4}$$

$$= \frac{4n + 14n(n-1)}{4} + \frac{(8 + n - 3)\{n(n-1)(n-2)\}}{4}$$

$$= \frac{4n + \{14 + (n+5)(n-2)\}\,n(n-1)}{4}$$

$$= \frac{4n + n^4 + 2n^3 + n^2 - 4n}{4} = \frac{n^4 + 2n^3 + n^2}{4}$$

$$= \frac{n^2(n+1)^2}{4} = \text{sum.}$$

Ex. 7. 1, 16, 81, 256, 625, 1296, 2401,

15, 65, 175, 369, 671, 1105,

50, 110, 194, 302, 434,

60, 84, 108, 132,

24, 24, 24,

0, 0.

Here $\Delta^1 = 15$, $\Delta^2 = 50$, $\Delta^3 = 60$, $\Delta^4 = 24$, $\Delta^5 = 0$, and $a = 1$;

$$\therefore na + \frac{n(n-1)}{2}\Delta^1 + \frac{n(n-1)(n-2)}{2.3}\Delta^2 + \frac{n(n-1)(n-2)(n-3)}{2.3.4}\Delta^3$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{2.3.4.5}\Delta^4 =$$

$$n + \frac{15n^2 - 15n}{2} + \frac{25n^3 - 75n^2 + 50n}{3} + \frac{5n^4 - 30n^3 + 55n^2 - 30n}{2}$$

$$+ \frac{n^5 - 10n^4 + 35n^3 - 50n^2 + 24n}{5};$$

and by reducing these fractions to the common denominator 30 (which is the least multiple), and then collecting the terms of the numerator, the preceding will become

$$\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} = \text{sum}.$$

APPLICATION OF FORMULA II.

Page 216.

Ex. 3. Here $p=2$, $q=1, 2, 3$, &c. and $n=1, 3, 5$, &c. successively;

$$\therefore \left\{\begin{array}{l} \frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+\text{\&c.} \\ \quad -(\frac{1}{3.5}+\frac{2}{5.7}+\text{\&c.}) \end{array}\right\} = \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\text{\&c.}=\frac{1}{2}$$

(Alg. p. 212);

$$\text{and } \frac{1}{2p} \text{ of } \frac{1}{2}=\frac{1}{8}=\text{sum.}$$

Ex. 4. Here $p=2$, $q=1, 4, 7$, &c., and $n=1, 3, 5$, &c., successively;

$$\therefore \left\{\begin{array}{l} \frac{1}{1.3}+\frac{4}{3.5}+\frac{7}{5.7}+\frac{10}{7.9}+\text{\&c.} \\ \quad -(\frac{1}{3.5}+\frac{4}{5.7}+\frac{7}{7.9}+\text{\&c.}) \end{array}\right\} = \frac{1}{1.3}+\frac{3}{3.5}+\frac{3}{5.7}+\frac{3}{7.9}+\text{\&c.}$$

$$=\frac{1}{1.3}+\frac{1}{2}\left\{\begin{array}{l} 1+\frac{3}{5}+\frac{3}{7}+\text{\&c.} \\ \quad -(\frac{3}{5}+\frac{3}{7}+\text{\&c.}) \end{array}\right\} =$$

$$\frac{1}{3}+\frac{1}{2}=\frac{5}{6};\ \text{and } \frac{1}{2p} \text{ of this is } \frac{5}{24}=\text{sum.}$$

Ex. 5. Here,

$$\left\{\begin{array}{l} \frac{a}{n(n+p)}+\frac{a+b}{(n+p)(n+2p)}+\frac{a+2b}{(n+2p)(n+3p)}+\text{\&c.} \\ -(\frac{a}{(n+p)(n+2p)}+\frac{a+b}{(n+2p)(n+3p)}+\frac{a+2b}{(n+3p)(n+4p)}+\text{\&c.}) \end{array}\right\}$$

$$=\frac{a}{n(n+p)}+\frac{b}{(n+p)(n+2p)}+\frac{b}{(n+2p)(n+3p)}+\frac{b}{(n+3p)(n+4p)}$$
$$+\ \text{\&c.}$$

$$= \frac{a}{n(n+p)} + \frac{1}{p}\left\{\begin{array}{l} \frac{b}{n+p} + \frac{b}{n+2p} + \frac{b}{n+3p} + \&c. \\ \quad -\left(\frac{b}{n+2p} + \frac{b}{n+3p} + \frac{b}{n+4p} + \&c.\right) \end{array}\right\} =$$

$$\frac{a}{n(n+p)} + \frac{1}{p}\left(\frac{b}{n+p}\right) = \frac{pa\,(n+p) + bn\,(n+p)}{pn\,(n+p)^2} = \frac{pa+bn}{pn\,(n+p)};$$

and $\frac{1}{2p}$ of this is $\frac{pa+bn}{2p^2n\,(n+p)} = \text{sum}.$

APPLICATION OF FORMULA III.

Page 217.

Ex. 3. Here $p = 3$,

$$\therefore \left\{\begin{array}{l} \frac{2}{3.6.9} + \frac{5}{6.9.12} + \frac{8}{9.12.15} + \&c. \\ \quad -\left(\frac{2}{6.9.12} + \frac{5}{9.12.15} + \frac{8}{12.15.18} + \&c.\right) \end{array}\right\}$$

$$= \frac{2}{3.6.9} + \frac{3}{6.9.12} + \frac{3}{9.12.15} + \&c. =$$

$$\frac{2}{3.6.9} + \frac{1}{2p}\left\{\begin{array}{l} \frac{3}{6.9} + \frac{3}{9.12} + \&c. \\ \quad -\left(\frac{3}{9.12} + \frac{3}{12.15} + \&c.\right) \end{array}\right\} =$$

$$\frac{2}{3.6.9} + \frac{1}{2p}\left(\frac{3}{6.9}\right) = \frac{2}{3.6.9} + \frac{1}{2.6.9} = \frac{7}{2.3.6.9};$$

and $\frac{1}{3p}$ of this is $\frac{7}{2916} = \text{sum}.$

Ex. 4. Here $p = 1$,

$$\left\{\begin{array}{l} \frac{36}{1.2.3} + \frac{49}{2.3.4} + \frac{64}{3.4.5} + \&c. \\ \qquad -(\frac{36}{2.3.4} + \frac{49}{3.4.5} + \frac{64}{4.5.6} + \&c.) \end{array}\right\}$$

$$= \frac{36}{1.2.3} + \frac{13}{2.3.4} + \frac{15}{3.4.5} + \&c. =$$

$$\frac{36}{1.2.3} + \frac{1}{2p}\left\{\begin{array}{l} \frac{13}{2.3} + \frac{15}{3.4} + \&c. \\ \qquad -(\frac{13}{3.4} + \frac{15}{4.5} + \&c.) \end{array}\right\} =$$

$$\frac{36}{6} + \frac{13}{12} + \frac{1}{2p}\left\{\frac{2}{3.4} + \frac{2}{4.5} + \&c. = \frac{2}{3}\right\} =$$

$$\frac{36}{6} + \frac{13}{12} + \frac{2}{6} = \frac{89}{12}; \text{ and } \frac{1}{3p} \text{ of this is } \frac{89}{36} = \text{sum.}$$

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Ex. 3. Here $a = 2$, $b = 2$, and $n = 3$.

$$\therefore \left\{\begin{array}{l} 2 + \frac{2.4}{3} + \frac{2.4.6}{3.5} + \ldots \frac{2.4.6.8.....(2r)}{3.5.7...(2r-1)} \\ -(\frac{2.4}{3} + \frac{2.4.6}{3.5} + \ldots\ldots\ldots\ldots \frac{2.4.6.8..(2r+2)}{3.5.7...(2r+1)}) \end{array}\right\}$$

$$= 2 - \frac{2.4.6.8 \ldots\ldots (2r+2)}{3.5.7.9 \ldots\ldots (2r+1)};$$

$$\text{and } \frac{1}{n-a-b} \text{ of this is } \frac{2.4.6.8 \ldots\ldots (2r+2)}{3.5.7.9 \ldots\ldots (2r+1)} - 2 = \text{sum.}$$

Ex. 4. To render this series suitable for the application of the general formula, we must write it in the form $\frac{1.2}{5.6} + \frac{1.2.3}{5.6.7} + \frac{1.2.3.4}{5.6.7.8} + \&c$; hence $a = 1$, $b = 1$, and $n = 5$;

$$\therefore \left\{\begin{array}{l} \frac{1.2}{5} + \frac{1.2.3}{5.6} + \frac{1.2.3.4}{5.6.7} + \&c. \\ \quad -\left(\frac{1.2.3}{5.6} + \frac{1.2.3.4}{5.6.7} + \&c.\right) \end{array}\right\} = \frac{2}{5};$$

$$\text{and } \frac{1}{n-a-b} \text{ of this is } \frac{2}{15} = \text{sum.}$$

PROMISCUOUS EXAMPLES.

Page 223.

Ex. 2. Here $p = 1$.

$$\left\{\begin{array}{l} \frac{5}{1.2.2^2} + \frac{6}{2.3.2^3} + \frac{7}{3.4.2^4} + \&c. \\ \qquad -\left(\frac{5}{2.3.2^2} + \frac{6}{3.4.2^3} + \&c.\right) \end{array}\right\}$$

$$= \frac{5}{1.2.2^2} - \left(\frac{4}{2.3.2^3} + \frac{5}{3.4.2^4} + \&c.\right);$$

$$\text{and } \frac{1}{2p} \text{ of this is}$$

$$\frac{5}{2.2^3} - \frac{1}{2}\left\{\frac{4}{2.3.2^3} + \frac{5}{3.4.2^4} + \&c.\right\};$$

hence, summing the series within the brackets, we have

$$\left\{\begin{array}{l} \frac{4}{2.2^3} + \frac{5}{3.2^4} + \&c. \\ \qquad -\left(\frac{4}{3.2^3} + \frac{5}{4.2^4} + \&c.\right) \end{array}\right\},$$

$$= \frac{4}{2.2^3} - \frac{3}{3.2^4} - \frac{4}{4.2^5} - \&c.;$$

$$\text{and } \frac{1}{2} \text{ of this is } \frac{2}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} - \&c.;$$

$$\text{but } -\frac{1}{2^5}-\frac{1}{2^6}-\frac{1}{2^7}-\&c.=\frac{1}{2^3}\left(-\frac{1}{2^2}-\frac{1}{2^3}-\frac{1}{2^4}-\&c.\right)$$

$$=\frac{1}{2^3}\left(-\frac{1}{2}\right) \text{ or } -\frac{1}{2.2^3}; \textit{ Art. 75.}$$

$$\therefore\ \frac{5}{2.2^3}-\left(\frac{2}{2^4}-\frac{1}{2.2^3}\right)=\frac{1}{4}=\text{sum.}$$

Ex. 3. Assume the series $=\dfrac{z}{(1-x)^3}=\dfrac{z}{1-3x+3x^2-x^3}$;

$$\begin{array}{l} \text{then, } x+3x^2+6x^3+10x^4+\&c. \\ \qquad 1-3x+3x^2-x^3 \\ \hline \qquad x+3x^2+6x^3+10x^4+\&c. \\ \qquad\quad -3x^2-9x^3-18x^4+\&c. \\ \qquad\qquad\qquad 3x^3+9x^4+\&c. \\ \qquad\qquad\qquad\qquad -x^4+\&c. \\ \hline \therefore\ z=x \quad * \quad * \quad * \quad * \\ \hline \end{array}$$

$$\text{hence } x+3x^2+6x^3+10x^4+\&c.=\frac{x}{(1-x)^3}=\text{sum.}$$

Ex. 4. The answer to this question will differ according as n is even or odd. By supposing n even, the operation will be as follows:

$$\left\{\begin{array}{l}\frac{1}{4}-\frac{1}{6}+\frac{1}{8}-\frac{1}{10}+\ldots-\frac{1}{2n+2} \\ \qquad -\left(\frac{1}{8}-\frac{1}{10}+\ldots-\frac{1}{2n+2}+\frac{1}{2n+4}-\frac{1}{2n+6}\right)\end{array}\right\}=$$

$$\frac{1}{4}-\frac{1}{6}-\frac{1}{2n+4}+\frac{1}{2n+6}=\left(\frac{1}{4}-\frac{1}{2n+4}\right)+\left(\frac{1}{2n+6}-\frac{1}{6}\right)$$

$$=\frac{2n}{8n+16}-\frac{2n}{12n+36}; \text{ and } \frac{1}{p}\text{th, that is, } \frac{1}{4}, \text{ of this is}$$

$$\frac{n}{16(n+2)} - \frac{n}{24(n+3)} \text{ Ans.}$$

Again, let n be odd, then the process will be

$$\left\{\begin{array}{l} \frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \frac{1}{10} + \ldots\ldots + \frac{1}{2n+2} \\ \quad - (\frac{1}{8} - \frac{1}{10} + \ldots\ldots + \frac{1}{2n+2} - \frac{1}{2n+4} + \frac{1}{2n+6}) \end{array}\right\} =$$

$$\frac{1}{4} - \frac{1}{6} + \frac{1}{2n+4} - \frac{1}{2n+6} = (\frac{1}{4} + \frac{1}{2n+4}) - (\frac{1}{6} + \frac{1}{2n+6}) =$$

$$\frac{2n+8}{8n+16} - \frac{2n+12}{12n+36}; \text{ and } \frac{1}{p}\text{th of this is}$$

$$\frac{n+4}{16(n+2)} - \frac{n+6}{24(n+3)} \text{ Ans.}$$

Ex. 5. Here the series may be written

$$\frac{1}{2.3}\left\{\frac{1}{4.6} + \frac{1}{5.7} + \frac{1}{6.8} + \&c. \right\};$$

$$\text{Then, } \frac{1}{6}\left\{\begin{array}{l} \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \&c. \\ \quad - (\frac{1}{6} + \frac{1}{7} + \&c.) \end{array}\right\} = \frac{1}{6}\left(\frac{9}{20}\right) = \frac{9}{120};$$

$$\text{and } \frac{1}{p}, \text{ that is } \frac{1}{2}, \text{ of this is } \frac{9}{240} = \frac{3}{80} = \text{sum.}$$

Ex. 6. Here the series may be written

$$\frac{2.3}{2.2.3.3}\left\{\frac{5.6}{1.2.3.4} + \frac{6.7}{2.3.4.5} + \frac{7.8}{3.4.5.6} + \&c. \right\} =$$

$$\frac{1}{3p} \text{ of } \frac{1}{6}\left\{\begin{array}{l} \frac{5.6}{1.2.3} + \frac{6.7}{2.3.4} + \frac{7.8}{3.4.5} + \&c. \\ \quad - (\frac{5.6}{2.3.4} + \frac{6.7}{3.4.5} + \&c.) \end{array}\right\} =$$

$$\frac{1}{18}\left\{\frac{30}{1.2.3}+\frac{12}{2.3.4}+\frac{14}{3.4.5}+\&c.\right\};$$

Then,

$$\frac{1}{18}\left\{\begin{array}{l}\frac{30}{1.2}+\frac{12}{2.3}+\frac{14}{3.4}+\frac{16}{4.5}+\&c.\\ -(\frac{30}{2.3}+\frac{12}{3.4}+\frac{14}{4.5}+\&c.)\end{array}\right\}=$$

$$\frac{1}{18}\left\{15-\frac{18}{2.3}+\frac{2}{3.4}+\frac{2}{4.5}+\&c.\right\};$$

and $\frac{1}{2p}$ of this is

$$\frac{1}{36}\left\{15-\frac{18}{6}+\frac{2}{3}\right\}=\frac{15}{36}-\frac{3}{36}+\frac{2}{108}=\frac{38}{108}=\frac{19}{54}=\text{sum}.$$

ON RECURRING SERIES.

PROBLEM I.

Page 227.

To find the sum of an infinite recurring series.

Ex. 2. Here the third term, $C=x^2A+xB$, whence

$$-x^2A-xB+C=0;$$

hence, $s=-x^2$, $r=-x$, $q=1$, and $p=0$;

$$\therefore\ \text{sum}=\frac{A(p+q+r)+B(p+q)+Cp}{p+q+r+s}=\frac{A(1-x)+B}{1-x-x^2}$$

$$=\frac{1+x}{1-x-x^2}.$$

Ex. 3. Here the third term, $C = -x^2A + 2xB$,

$$\text{whence, } x^2A - 2xB + C = 0;$$

$$\text{consequently, } s = x^2,\ r = -2x,\ q = 1, \text{ and } p = 0;$$

$$\therefore \text{ sum} = \frac{A(1-2x)+B}{1-2x+x^2} = \frac{1+x}{1-2x+x^2}.$$

Ex. 4. Here the fourth term, $D = -2x^3A + x^2B + 2xC$, whence

$$2x^3A - x^2B - 2xC + D = 0;$$

$$\text{hence, } s = 2x^3,\ r = -x^2,\ q = -2x, \text{ and } p = 1;$$

$$\therefore \text{ sum} = \frac{A(1-2x-x^2)+B(1-2x)+C}{1-2x-x^2+2x^3} = \frac{3-x-6x^2}{1-2x-x^2+2x^3}.$$

PROBLEM II.

Page 229.

To find the sum of any number of terms of a recurring series.

Ex. 2. Here $C = -x^2A + 2xB$, whence $x^2A - 2xB + C = 0$;

$$\therefore\ s = x^2,\ r = -2x,\ q = 1,\ p = 0, \text{ also } U = (2n+1)x^n;$$

$$V = (2n+3)x^{n+1}, \text{ and } W = (2n+5)x^{n+2};$$

$$\text{hence, sum} = \frac{(A-U)(1-2x)+B-V}{1-2x+x^2} =$$

$$\frac{\{1-(2n+1)x^n\}(1-2x)+3x-(2n+3)x^{n+1}}{1-2x+x^2}$$

$$= \frac{1-(2n+1)x^n+x+2(2n+1)x^{n+1}-(2n+3)x^{n+1}}{1-2x+x^2}$$

$$= \frac{1+x-(2n+1)x^n+(2n-1)x^{n+1}}{1-2x+x^2} = \text{sum}.$$

ON THE METHOD OF INDETERMINATE COEFFICIENTS.

Page 232.

Ex. 4. Assume $\sqrt{1-x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$;

then, by squaring each side, and transposing, we have

$$\left.\begin{matrix} A^2 \\ -1 \end{matrix}\right. \left.\begin{matrix} +2AB \\ 1 \end{matrix}\right\} x \left.\begin{matrix} +2AC \\ +B^2 \end{matrix}\right\} x^2 \left.\begin{matrix} +2AD \\ +2BC \end{matrix}\right\} x^3 \left.\begin{matrix} +2AE \\ +2BD \\ +C^2 \end{matrix}\right\} x^4 + \&c. = 0;$$

$$\text{Whence} \left\{ \begin{array}{lll} A^2 - 1 & = 0, \text{ therefore } A = & 1 \\ 2AB + 1 & = 0, \quad \ldots \quad B = & -\frac{1}{2} \\ 2AC + B^2 & = 0, \quad \ldots \quad C = & -\frac{1}{2.4} \\ 2AD + 2BC & = 0, \quad \ldots \quad D = & -\frac{3}{2.4.6} \\ 2AE + 2BD + C^2 & = 0, \quad \ldots \quad E = & -\frac{3.5}{2.4.6.8} \\ \&c. & & \&c. \end{array} \right.$$

$$\therefore \sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{2.4} - \frac{3x^3}{2.4.6} - \frac{3.5x^4}{2.4.6.8} - \&c.$$

Ex. 5. Assume $\dfrac{1+2x}{1-x-x^2} = A + Bx + Cx^2 + Dx^3 + \&c.$;

multiplying each side by $1 - x - x^2$, and transposing, we have

$$\left.\begin{matrix} A \\ -1 \end{matrix}\right. \left.\begin{matrix} -A \\ +B \\ -2 \end{matrix}\right\} x \left.\begin{matrix} -A \\ -B \\ +C \end{matrix}\right\} x^2 \left.\begin{matrix} -B \\ -C \\ +D \end{matrix}\right\} x^3 \left.\begin{matrix} -C \\ -D \\ +E \end{matrix}\right\} x^4 + \&c. = 0;$$

$$\text{Whence}\left\{\begin{array}{ll} A-1=0, \text{ therefore, } & A=1 \\ -A+B-2=0, \quad \ldots & B=3 \\ -A-B+C=0, \quad \ldots & C=4 \\ -B-C+D=0, \quad \ldots & D=7 \\ -C-D+E=0, \quad \ldots & E=11 \\ \&c. & \&c. \end{array}\right.$$

$$\therefore \frac{1+2x}{1-x-x^2}=1+3x+4x^2+7x^3+11x^4+\&c.$$

Ex. 6. Assume $\dfrac{1}{1-2ax+x^2}=A+Bx+Cx^2+Dx^3+\&c.$;

then, multiplying each side by $1-2ax+x^2$, and transposing, we have

$$\left.\begin{array}{r} A \\ -1 \end{array}\right. \left.\begin{array}{r} +B \\ -2Aa \end{array}\right\} x \left.\begin{array}{r} +C \\ -2Ba \\ +A \end{array}\right\} x^2 \left.\begin{array}{r} +D \\ -2Ca \\ +B \end{array}\right\} x^3 \left.\begin{array}{r} +E \\ -2Da \\ +C \end{array}\right\} x^4+\&c.,=0;$$

$$\text{Whence}\left\{\begin{array}{ll} A-1=0, \text{ therefore, } & A=1 \\ B-2Aa=0, \quad \ldots & B=2a \\ C-2Ba+A=0, \quad \ldots & C=4a^2-1 \\ D-2Ca+B=0, \quad \ldots & D=8a^3-4a \\ E-2Da+C=0, \quad \ldots & E=16a^4-12a^2+1. \end{array}\right.$$

$$\therefore \frac{1}{1-2ax+x^2}=1+2ax+(4a^2-1)x^2+(8a^3-4a)x^3+$$

$$(16a^4-12a^2+1)x^4+\&c.$$

ON THE MULTINOMIAL THEOREM.

Page 236.

Ex. 3. Here $a = 2$, $b = 3$, $c = 4$, $d = 5$, &c., and $Q = 12$, therefore

$$a^n = 8 = \mathrm{A}$$

$$Qb = 36 = \mathrm{B}$$

$$\frac{(n-1)\,\mathrm{B}b}{2a} + Qc = 102 = \mathrm{C}$$

$$\frac{(n-2)\,\mathrm{C}b + (2n-1)\,\mathrm{B}c}{3a} + Qd = 231 = \mathrm{D};$$

&c. &c.

$$\therefore (2x + 3x^2 + 4x^3 + \&c.)^3 = x^3\,(2 + 3x + 4x^2 + \&c.)^3 =$$

$$8x^3 + 36x^4 + 102x^5 + 231x^6 + \&c.$$

Ex. 4. Here $a = 1$, $b = -\frac{1}{3}$, $c = \frac{1}{5}$, $d = -\frac{1}{7}$, $e = \frac{1}{9}$, &c., and $Q = 2$, therefore,

$$a^n = 1 = \mathrm{A}$$

$$Qb = -\frac{2}{3} = \mathrm{B}$$

$$\frac{(n-1)\,\mathrm{B}b}{2a} + Qc = \frac{23}{45} = \mathrm{C}$$

$$\frac{(n-2)\,\mathrm{C}b + (2n-1)\,\mathrm{B}c}{3a} + Qd = -\frac{44}{105} = \mathrm{D}$$

$$\frac{(n-3)\,\mathrm{D}b + (2n-2)\,\mathrm{C}c + (3n-1)\,\mathrm{B}d}{4a} + Qe = \frac{563}{1575} = \mathrm{E};$$

&c. &c.

$$\therefore\ (1 - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \&c.)^2 = 1 - \frac{2}{3}x^3 + \frac{23}{45}x^5 - \frac{44}{105}x^7$$

$$+ \frac{563}{1575}x^9 - \&c.$$

Ex. 5. Here $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{3}$, $d = \frac{1}{4}$, $e = \frac{1}{5}$, &c. also

$Q = \frac{1}{3}$, therefore

$a^n = 1 = A$

$Qb = \frac{1}{6} = B$

$\frac{(n-1)\,Bb}{2a} + Qc = \frac{1}{12} = C$

$\frac{(n-2)\,Cb + (2n-1)\,Bc}{3a} + Qd = \frac{35}{648} = D$

$\frac{(n-3)\,Db + (2n-2)\,Cc + (3n-1)\,Bd}{4a} + Qe = \frac{383}{9720} = E$

&c. &c.

$$\therefore (1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \&c.)^{\frac{1}{3}} = 1 + \frac{1}{6}x + \frac{1}{12}x^2 + \frac{35}{648}x^3$$

$$+ \frac{383}{9720}x^4 + \&c.$$

ON THE REVERSION OF SERIES.

Page 239.

Ex. 3. Here $a=1$, $b=-\frac{1}{2}$, $c=\frac{1}{4}$, $d=-\frac{1}{8}$, &c.

Therefore, $\frac{1}{a}=1$,

$$-\frac{b}{a^3}=\frac{1}{2},$$

$$\frac{2b^2-ac}{a^5}=\frac{1}{4},$$

$$-\frac{5b^3-5abc+a^2d}{a^7}=\frac{1}{8};$$

&c. &c.

$$\therefore\ x=y+\frac{1}{2}y^2+\frac{1}{4}y^3+\frac{1}{8}y^4+\text{\&c.}$$

Ex. 4. Here $a=1$, $b=-\frac{1}{3}$, $c=\frac{1}{5}$, $d=-\frac{1}{7}$, &c.

Therefore, $\frac{1}{a}=1$,

$$-\frac{b}{a^4}=\frac{1}{3},$$

$$\frac{3b^2-ac}{a^7}=\frac{2}{15},$$

$$-\frac{12b^3+a^2d-8abc}{a^{10}}=\frac{17}{315};$$

&c. &c.

$$\therefore x = y + \frac{1}{3} y^3 + \frac{2}{15} y^5 + \frac{17}{315} y^7 + \&c.$$

Ex. 5. By transposition, we have

$$x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \&c. = y - 1;$$

$$\text{hence } a = 1,\ b = \frac{1}{2},\ c = \frac{1}{6},\ d = \frac{1}{24},\ \&c.$$

Therefore, $\frac{1}{a} = 1,$

$$-\frac{b}{a^3} = -\frac{1}{2},$$

$$\frac{2b^2 - ac}{a^5} = \frac{1}{3},$$

&c.

$$\therefore x = (y-1) - \frac{(y-1)^2}{2} + \frac{(y-1)^3}{3} - \&c.$$

ON INDETERMINATE EQUATIONS.

PROBLEM I.

To find the integer values of x and y in the equation

$$ax \pm by = c.$$

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Ex. 4. The expression for x deduced from this equation is

$$x = \frac{254 - 11y}{5} = 50 - 2y - \frac{y-4}{5} \quad . \quad . \quad . \quad (A).$$

Hence, proceeding according to the rule,

$$\begin{array}{l} y-4 \\ 4 \\ \hline 4y-16 \\ 5y \\ \hline \;\; y+16 \end{array}$$

and as $\frac{16}{5}$ gives a remainder $R=1$, therefore $5-1=4$ is the least value of y. Now, it is obvious, from a mere inspection of the proposed equation, that the less y is, the greater is x; hence by (A) the greatest value of x is 42; and by adding 5 (the coefficient of x) repeatedly to the least value of y, and subtracting 11 (the coefficient of y) from the greatest value of x, we shall have all the possible answers as follows, viz.

$$\begin{array}{r|c|c|c|} y=4 & 9 & 14 & 19 \\ x=42 & 31 & 20 & 9 \end{array}$$

Ex. 5. Here we shall determine the least value of x by means of the expression for y, which is

$$y=\frac{500-11x}{35}=14-\frac{11x-10}{35};$$

$$\begin{array}{rl} & 11x-10 \\ & 3 \\ \cline{2-2} & 33x-30 \\ & 35x \\ \cline{2-2} & \;\; 2x+30 \\ & \;\; 5 \\ \cline{2-2} \text{Subtracting} & 10x+150 \text{ from above} \\ \cline{2-2} \text{gives} & x-160 \end{array}$$

Now, $\frac{160}{35}$ gives R $=20=$ the least value of x.

Ex. 6. Here

$$x=\frac{11+117y}{19}=6y+\frac{3y+11}{19} \quad . \quad . \quad . \quad (A)$$

$$\begin{array}{l} 3y+11 \\ 6 \\ \hline 18y+66 \\ 19y \\ \hline y-66 \end{array}$$

and $\frac{66}{19}$ gives R $=9=$ the least value of y, and it is evident from (A) that the less y is the less will x be; hence the least value of x as given by (A) is 56.

Ex. 7. Let x represent the number of guineas, and y the number of three-shilling pieces; then, reducing all to shillings, we have to ascertain whether the equation

$$21x+3y=1000$$

be possible or impossible in positive integers.

For the value of x we have

$$x=\frac{1000-3y}{21}=47-\frac{3y-13}{21};$$

and it is plain that whatever multiples we take of 3 and 21, their difference must be either 0, or else some multiple of 3; hence it follows that the solution in integers is impossible.

Ex. 8. Let x be the number of sheep, and y that of the lambs; then, by the question,

$$26x + 15y = 168;$$

$$\therefore\ x = \frac{168 - 15y}{26} = 6 - \frac{15y - 12}{26} \quad . \ . \ . \quad (A)$$

$$\begin{array}{rl}
 & 15y - 12 \\
 & 2 \\ \hline
 & 30y - 24 \\
 & 26y \\ \hline
 & 4y - 24 \\
 & 4 \\ \hline
\text{Subtracting} & 16y - 96 \text{ from above} \\ \hline
\text{we have} & y - 84
\end{array}$$

and $\frac{84}{26}$ gives R $= 6$, the number of lambs;

$\therefore$ (A), 3 is that of the sheep.

Ex. 9. For x, we have the expression

$$x = \frac{71 - 13y}{7} = 10 - y - \frac{6y - 1}{7} \quad . \ . \ . \quad (A)$$

$$\begin{array}{l}
7y \\
6y - 1 \\ \hline
y + 1
\end{array}$$

$\therefore$ R $= 1$, and $7 - 1 = 6 =$ the least value of y; therefore, from (A), the greatest value of x is -1; hence, the equation admits of no solution in positive integers.

PROBLEM II.

Page 248.

Ex. 2. We have here to ascertain whether the relation

$$21x + 3y = 2800$$

is possible in positive integers.

The equation $21x' - 3y' = 1$ gives

$$x' = \frac{3y' + 1}{21};$$

which is impossible (see Solution to Ex. 7, page 155).

Ex. 3. Here, the proposed equation is

$$21x + 5y = 20000;$$

and from the equation

$$21x' - 5y' = 1,$$

we have

$$x' = \frac{5y' + 1}{21};$$

then,

$$\begin{array}{l} 5y' + 1 \\ 4 \\ \hline 20y' + 4 \\ 21y' \\ \hline y' - 4 \end{array}$$

$\therefore y' = 4$, and $x' = 1$; consequently, the number of solutions is the integral part of $\frac{20000 \times 1}{5}$ — integ. part of $\frac{20000 \times 4}{21} = 190$.

PROBLEM III.

Page 251.

Ex. 2. Here, z cannot be greater than $\dfrac{400-17-19}{21} = 17\frac{1}{3}$,

and by Problem I.,

$$x = \frac{400-19y-21z}{17} = 23 - y - z - \frac{2y+4z-9}{17} \quad . \; . \; . \quad (A)$$

$$\begin{array}{l} 2y + 4z - 9 \\ 8 \\ \hline 16y + 32z - 72 \\ 17y \\ \hline \quad y - 32z + 72 \end{array}$$

By putting $z = 1$, this remainder becomes $y + 40$, and $\dfrac{40}{17}$ gives $R = 6$, $\therefore 17 - 6 = 11$, the value of y corresponding to the least value of z; also the value of x from (A) is 10; by putting 2, 3, 4, &c. for z in the same remainder, according to the rule, we shall have all the answers as follow:

$z=1$	2	3	4	5	6	11	12	13	14
$y=11$	9	7	5	3	1	8	6	4	2
$x=10$	11	12	13	14	15	1	2	3	4

PROBLEM IV.

Page 254.

Ex. 2. Here the greatest limit of $x < \dfrac{100-3-5}{7}$ is 13; also in

the equation $3x' - 5y' = 1$, we have $a = 3$, $b = 5$, $x' = 2$, $y' = 1$; consequently, the two series of which the sums are required, beginning with the least terms, $\frac{(d-13c)x'}{b}$ and $\frac{(d-13c)y'}{a}$, will be

$$\frac{2.9}{5} + \frac{2.16}{5} + \frac{2.23}{5} + \; . \; . \; . \; . \frac{2.93}{5}$$

$$\text{and } \frac{1.9}{3} + \frac{1.16}{3} + \frac{1.23}{3} + \; . \; . \; . \; . \frac{1.93}{3},$$

the common difference of the terms in the first series being $\frac{2.7}{5}$, and in the second $\frac{7}{3}$; and the number of terms in each 13.

Hence the sum of the first series is $265\frac{1}{5}$, and that of the second series 221.

Moreover, the first period of fractions in the first series is

$$\frac{3}{5} + \frac{2}{5} + \frac{1}{5} + \frac{5}{5} + \frac{4}{5} = 3 \quad . \; . \; . \; . \quad (A),$$

and the first period in the second series is

$$0 + \frac{1}{3} + \frac{2}{3} = 1 \quad . \; . \; . \; . \quad (B);$$

and since the number of terms is 13 in each series, we must take two periods and three terms of (A) for the fractional parts of the first series, and four periods and one term of (B) for the fractional parts of the second series. These parts are respectively $7\frac{1}{5}$ and 4;

$$\therefore \; (265\tfrac{1}{5} - 7\tfrac{1}{5}) - (221 - 4) = 41. \text{ Ans.}$$

Ex. 3. The greatest limit of $z < \frac{9999 - 7 - 9}{23}$ is 434; and in the equation $7x' - 9y' = 1$, we have $a = 7$, $b = 9$, $x' = 4$, $y' = 3$; conse-

quently, the two series beginning with the least terms $\frac{(d-434c)x'}{b}$ and $\frac{(d-434c)y'}{a}$ are

$$\frac{4.17}{9}+\frac{4.40}{9}+\frac{4.63}{9}+\ldots\ldots\frac{4.9976}{9}$$

$$\frac{3.17}{7}+\frac{3.40}{7}+\frac{3.63}{7}+\ldots\ldots\frac{3.9976}{7}.$$

The number of terms in each series is 434; the sum of the first is $963769\frac{3}{9}$; and that of the second, 929349.

The first period of fractions in the first series is

$$\frac{5}{9}+\frac{7}{9}+\frac{9}{9}+\text{\&c. to nine terms}=\frac{45}{9}=5 \quad \ldots\ldots \quad \text{(A)}$$

and the first period of the second series is

$$\frac{2}{7}+\frac{1}{7}+0+\text{\&c. to seven terms}=\frac{21}{7}=3 \quad \ldots\ldots \quad \text{(B)}$$

hence, the number of terms being 434, we must take 48 periods and two terms of (A); and 62 periods of (B) for the fractional parts; these are respectively $241\frac{3}{9}$ and 186;

$$\therefore\ (963769\tfrac{3}{9}-241\tfrac{3}{9})-(929349-186)=34365.\ \text{Ans.}$$

PROBLEM V.

Page 256.

Ex. 2. Here, by multiplying the first equation by 3, then subtracting the result from the second, and afterwards dividing by 2, we shall have $5y+14z=620$;

$$\text{whence } y=\frac{620-14z}{5}=124-2z-\frac{4z}{5};$$

hence z must be 5, or some multiple of 5; and upon trial the numbers which fulfil the conditions of the question are

$$\begin{array}{l|l} z = 15 & 30 \\ y = 82 & 40 \\ x = 15 & 50. \end{array}$$

PROBLEM VI.

Page 258.

Ex. 2. Here $N = 17x + 7 = 26y + 13$;

$$\therefore 26y - 17x = -6, \text{ and } y = \frac{17x - 6}{26};$$

$$\begin{array}{r} 26x \\ 17x - 6 \\ \hline 9x + 6 \\ 2 \\ \hline 18x + 12 \\ 17x - 6 \\ \hline x + 18 \\ \hline \end{array}$$

Hence $26 - 18 = 8$ is the least value of x; and, consequently, the least whole number fulfilling the proposed conditions is

$$N = 17x + 7 = 143.$$

Ex. 3. Here $N = 28x + 19 = 19y + 15 = 15z + 11$;

$$\therefore 19y - 28x = 4, \text{ and } y = \frac{28x + 4}{19} = x + \frac{9x + 4}{19};$$

$$\begin{array}{l} 9x + 4 \\ 2 \\ \hline 18x + 8 \\ 19x \\ \hline \quad x - 8 \\ \hline \end{array}$$

Therefore the least value of x is 8; and, consequently, the least whole number fulfilling the conditions is

$$28x + 19 = 243.$$

Hence we have

$$N = 28 \times 19x' + 243 = 532x' + 243 = 15z + 11;$$

$$\therefore z = \frac{532x' + 232}{15} = 35x' + 15 + \frac{7x' + 7}{15};$$

$$\begin{array}{l} 7x' + 7 \\ 2 \\ \hline 14x' + 14 \\ 15x' \\ \hline \quad x' - 14 \\ \hline \end{array}$$

Consequently, the least value of x' is 14; and therefore

$$N = 532x' + 243 = 7691.$$

Ex. 4. Here $N = 3w + 2 = 5x + 4 = 7y + 6 = 2z;$

$$\therefore 5x - 3w = -2 \quad \therefore x = \frac{3w - 2}{5};$$

$$\begin{array}{l} 3w - 2 \\ 2 \\ \hline 6w - 4 \\ 5w \\ \hline w - 4 \end{array}$$

Hence the least value of w is 4; and therefore the least whole number fulfilling the first two conditions is

$$3w + 2 = 14.$$

Hence we have

$$N = 3 \times 5x' + 14 = 15x' + 14 = 7y + 6;$$

$$\therefore y = \frac{15x' + 8}{7} = 2x' + 1 + \frac{x' + 1}{7};$$

$$\begin{array}{l} x' + 1 \\ 6 \\ \hline 6x' + 6 \\ 7x' \\ \hline x' - 6 \\ \hline \end{array}$$

Consequently, the least value of x' is 6; and hence, the least whole number fulfilling the first three conditions is

$$15x' + 14 = 104;$$

and this an even number, it also fulfills the last condition; therefore, the required number is 104.

Ex. 5. In order that a number may be divisible by the nine digits, it is obviously sufficient that it be divisible by 5, 7, 8, and 9; and it is

evident that no number can be thus divisible if it be smaller than the product of these factors; hence, the least number is

$$5 \times 7 \times 8 \times 9 = 2520.$$

ON THE DIOPHANTINE ANALYSIS.

As some very elegant solutions of several of the Diophantine questions have already been given by Mr. Ward (in the American edition of Young's Algebra, which he so ably edited,) and many other mathematicians, I have occasionally preferred introducing a few of them here, to the unnecessary trouble of undertaking new solutions.

PROBLEM I.

To find such values of x as will render rational the expression

$$\sqrt{ax^2 + bx + c}.$$

CASE 1. *When $a = 0$, or when the expression is of the form $\sqrt{bx + c}$.*

Page 260.

Ex. 3. Let x be the number; then, as this number multiplied by 9, and the product diminished by 7, is to be a square, we will put

$$9x - 7 = p^2;$$

$$\therefore x = \frac{p^2 + 7}{9};$$

where p may be any number whatever.

Ex. 4. Let the number be x; then put

$$x + \tfrac{1}{5}x - 11 = p^2;$$

$$\therefore \tfrac{6}{5}x = p^2 + 11,$$

$$\text{whence } x = \frac{5(p^2 + 11)}{6};$$

p being any number whatever.

Case 2. *When $c = 0$, or when the expression is of the form*

$$\sqrt{ax^2 + bx}.$$

Page 261.

Ex. 3. Let x be the number, and assume

$$7x^2 + 8x = p^2x^2;$$

$$7x + 8 = p^2x,$$

$$p^2x - 7x = 8$$

$$\therefore x = \frac{8}{p^2 - 7};$$

where p may be taken at pleasure.

Ex. 4. Let x be the number; and make

$$\frac{x^2}{10} - \frac{x}{3} = p^2x^2,$$

$$3x^2 - 10x = 30p^2x^2;$$

$$3x - 10 = 30p^2x;$$

$$(3 - 30p^2)x = 10;$$

$$\therefore x = \frac{10}{3 - 30p^2}.$$

p being any value whatever.

CASE 3. *When c is a square, or when the expression is of the form*

$$\sqrt{ax^2 + bx + c^2}.$$

Page 261.

Ex. 2. Let one number be x, and the other $x + a$; put then

$$x^2 + (x + a)^2 = (px + a)^2;$$

$$\text{that is, } 2x^2 + 2ax + a^2 = p^2x^2 + 2apx + a^2;$$

$$p^2x - 2x = 2a\,(1 - p);$$

$$\therefore\ x = \frac{2a\,(1-p)}{p^2 - 2}:$$

where p may be any number we please.

CASE 4. *When a is a square, or when the expression is of the form*

$$\sqrt{a^2x^2 + bx + c}.$$

Page 262.

Ex. 2. Let x be one number, and $x + 14$ the other; then, since by the question $(x + 3)\,(x + 18)$ is to be a square, let us put

$$(x + 3)\,(x + 18) = (x + p)^2;$$

$$x^2 + 21x + 54 = x^2 + 2px + p^2,$$

$$(21 - 2p)\,x = p^2 - 54;$$

$$\therefore\ x = \frac{p^2 - 54}{21 - 2p}:$$

p being any value whatever.

Ex. 3. Put x for the first number, and $x + 3$ for the second; then,

by the question, $(2x + 3)(2x + 3)$ must be a square, which it obviously is for every value of x. Hence any two numbers whose difference is 3, will satisfy the question.

Case 5. *When neither a nor c is a square, but when $b^2 - 4ac$ is a square.*

Page 264.

Ex. 2. Here $a = 2$, $b = 10$, and $c = 12$; and $b^2 - 4ac$ is a square, viz. 4; hence the expression may be decomposed into two factors, which are $(2x + 6)$ and $(x + 2)$. Putting, then,

$$2x^2 + 10x + 12, \text{ or } (2x + 6)(x + 2) = p^2 (x + 2)^2$$
$$\text{we have } 2x + 6 = p^2 (x + 2);$$
$$\text{whence } x = \frac{6 - 2p^2}{p^2 - 2};$$

where p may be any assumed number.

Ex. 3. Here $b^2 - 4ac = 100$ is a square, and the expression $8x^2 + 6x - 2$ may be decomposed into $(8x - 2)$ and $x + 1$; putting, therefore,

$$(8x - 2)(x + 1) = p^2 (8x - 2)^2;$$
$$\text{we obtain } x + 1 = p^2 (8x - 2);$$
$$\therefore\ x = \frac{2p^2 + 1}{8p^2 - 1};$$

where p may be assumed at pleasure.

Case 6. *When the proposed expression can be divided into two parts, one of which is a square, and the other the product of two factors.*

Page 265.

Ex. 2. Here the expression $12x^2 + 17x + 6$ is found equivalent to

$$(3x + 2)^2 + (x + 1)(3x + 2);$$

and equating this with

$\{(3x+2) - p(x+1)\}^2$ or $(3x+2)^2 - 2p(3x+2)(x+1) + p^2(x+1)^2$,

we obtain

$$3x + 2 = -2p(3x + 2) + p^2(x + 1);$$

$$\therefore\ (6p - p^2 + 3)\,x = p^2 - 4p - 2;$$

$$\text{whence } x = \frac{p^2 - 4p - 2}{6p - p^2 + 3};$$

where p may be taken at pleasure.

Additional Examples.

Page 266.

Ex. 2. The expression $\sqrt{5x^2 + 12x + 8}$ is readily found to be rational when $x = 1$; putting, then, $x = 1 + y$, we have, by substituting,

$$5x^2 + 12x + 8 = 25 + 22y + 5y^2,$$

which must be a square. Let it be represented by

$$(5 + py)^2 = 25 + 10py + p^2y^2;$$

$$\text{hence } 25 + 22y + 5y^2 = 25 + 10py + p^2y^2;$$

$$\text{or } 22 + 5y = 10p + p^2y;$$

$$\text{Consequently, } y = \frac{22 - 10p}{p^2 - 5};$$

where p may be assumed at pleasure. If we take $p = 2$, then $y = -2$, and $x = -1$, for which value the proposed expression is 1. If $p = 1$, then $y = -3$, and $x = -2$, and the proposed expression becomes 2, and so on.

Ex. 3. Let x be the required number, then we shall have to render the expression $\sqrt{3x^2 - 3x + 3}$ rational. We readily see that one suit-

able value is $x = 2$; putting, therefore, $x = 2 + y$, we have, by substitution,

$$9 + 9y + 3y^2 = \square \quad . \; . \; . \; . \text{ (a square)}$$

assume it equal to

$$(3 + py)^2 = 9 + 6py + p^2y^2,$$

then we have

$$\begin{aligned} 9y + 3y^2 &= 6py + p^2y^2; \\ 9 + 3y &= 6p + p^2y; \\ (p^2 - 3)y &= 9 - 6p; \\ \therefore\ y &= \frac{9 - 6p}{p^2 - 3}; \end{aligned}$$

p being as usual any number we please.

$$\begin{array}{llll} \text{If we take } p = 0, & \text{then } y = -3, & \text{and } x = -1, \\ p = 1, & \ldots\ y = -\frac{3}{2}, & \ldots\ x = \frac{1}{2}, \\ p = 2, & \ldots\ y = -3, & \ldots\ x = -1, \\ \&c. & \&c. & \&c. \end{array}$$

and all these values of x are found upon trial to render the expression rational.

Or thus: (by Case VI.)

Ex. 3. Let x be the required number, then we shall have

$$3x^2 - 3x + 3 = \square;$$

which may be decomposed into

$$(x - 2)^2 + (2x - 1)(x + 1),$$

and equating this with

$$[(x - 2) + p(x + 1)]^2 = (x - 2)^2 + 2p(x + 1)(x - 2) + p^2(x + 1)^2$$

$$\text{gives } 2x - 1 = 2p(x-2) + p^2(x+1);$$

$$\text{whence } x = \frac{p^2 - 4p + 1}{2 - 2p - p^2}.$$

By taking $p = 2$, we obtain $x = \frac{1}{6}$.

PROBLEM II.

To find such values of x as will render rational the expression

$$\sqrt{ax^3 + bx^2 + cx + d}.$$

CASE 1. *When the last two terms are absent, or when the expression is of the form*

$$\sqrt{ax^3 + bx^2}.$$

Page 266.

Ex. 2. Let x be the number; then we must have

$$3x^3 - 5x^2 = \square,$$

which square we will denote by p^2x^2; hence

$$3x - 5 = p^2,$$

$$\therefore\ x = \frac{p^2 + 5}{3}.$$

By taking $p = 2$, we find $x = 3$.

CASE 2. *When the last term is a square, or when the expression is of the form*

$$\sqrt{ax^3 + bx^2 + cx + d^2}.$$

Page 268.

Ex. 2. Put $x^3 - x^2 + 2x + 1 = (x+1)^2 = x^2 + 2x + 1$;

$$\text{hence } x^3 = 2x^2;$$

$$\therefore x = 2.$$

Ex. 3. Put, here, $-5x^3 + 4x^2 - 6x + 1 = (-3x + 1)^2 = 9x^2 - 6x + 1$ and we shall have $-5x^3 = 9x^2 - 4x^2$, or $-5x = 5$;

$$\text{whence } x = \frac{5}{-5} = -1.$$

Additional Examples.

Page 269.

Ex. 2. Put $x = y + 1$, then we shall have

$$x^3 + 3 = y^3 + 3y^2 + 3y + 4,$$

which last assume equal to

$$(\frac{3}{4}y + 2)^2, \text{ or } \frac{9}{16}y^2 + 3y + 4;$$

$$\therefore y^3 + 3y^2 = \frac{9}{16}y^2, \text{ or } y + 3 = \frac{9}{16};$$

$$\text{hence } y = -\frac{39}{16}; \text{ and } x = -\frac{23}{16}.$$

Ex. 3. Put, here, $x = y + 1$; then

$$3x^3 + 1 = 3y^3 + 9y^2 + 9y + 4;$$

assume this last expression equal to

$$(\frac{9}{4}y + 2)^2, \text{ or } \frac{81}{16}y^2 + 9y + 4;$$

$$\text{then } 3y^3 + 9y^2 = \frac{81}{16}y^2, \text{ or } 3y + 9 = \frac{81}{16}, \text{ or } 3y = \frac{81 - 144}{16}.$$

$$\text{Whence } y = -\frac{21}{16}, \text{ and } \therefore x = -\frac{5}{16}.$$

PROBLEM III.

To find such values of x as will render rational the expression

$$\sqrt{ax^4 + bx^3 + cx^2 + dx + e}.$$

CASE I. *When both the first and last terms are complete squares, or when the expression is of the form*

$$\sqrt{a^2x^4 + bx^3 + cx^2 + dx + e^2}.$$

Page 271.

Ex. 2. Put, by the first method,

$$x^4 - 2x^3 + 2x^2 + 2x + 1 = (x^2 - x + 1)^2$$
$$= x^4 - 2x^3 + 3x^2 - 2x + 1;$$

and there results

$$2x^2 + 2x = 3x^2 - 2x;$$
$$2x + 2 = 3x - 2;$$
$$x = 4;$$

by putting, according to the second method, the expression equal to

$$(x^2 + x + 1)^2 = x^4 + 2x^3 + 3x^2 + 2x + 1,$$
$$\text{we find } -2x^3 + 2x^2 = 2x^3 + 3x^2;$$
$$\text{whence } x = -\tfrac{1}{4}.$$

Ex. 3. Here $b = 0$; put, therefore, by the first method,

$$4x^4 + 3x + 1 = (2x^2 + 1)^2 = 4x^4 + 4x^2 + 1;$$
$$3x = 4x^2;$$
$$\therefore\ x = \tfrac{3}{4}.$$

Or, according to the second method, put

$$4x^4 + 3x + 1 = (2x^2 + \tfrac{3}{2}x + 1)^2 = 4x^4 + 6x^3 + \tfrac{25}{4}x^2 + 3x + 1;$$

$$\text{whence } 6x^3 + \frac{25}{4}x^2 = 0;$$

$$\therefore\ x = -\frac{25}{24}.$$

It appears, therefore, that the two values of x, as given by this method are $\frac{4}{3}$ and $-\frac{25}{24}$, and not $\frac{278}{576}$, $\frac{417}{576}$, as in the book. We may further observe that, since the first term in the proposed equation is a square, a solution will be immediately obtained, by equating the remaining terms to zero, thus:

$$3x + 1 = 0;\ \therefore x = -\tfrac{1}{3};$$

which value makes the proposed expression a square.

Case 2. *When the first term only is a square, or when the expression is of the form*

$$\sqrt{a^2x^4 + bx^3 + cx^2 + dx + e}.$$

Page 272.

Ex. 2. Here $m = 0$, and $n = 0$; therefore, put $x^4 - 3x + 2 = (x^2)^2 = x^4$;

$$\text{hence } -3x + 2 = 0;$$

$$\therefore\ x = \tfrac{2}{3}.$$

Ex. 3. Here $m = -1$, and $n = \frac{3}{2}$; therefore

$$\text{put } x^4 - 2x^3 + 4x^2 - 2x + 2 = (x^2 - x + \tfrac{3}{2})^2 =$$

$$x^4 - 2x^3 + 4x^2 - 3x + \tfrac{9}{4},$$

$$\text{and we have } -2x + 2 = -3x + \tfrac{9}{4};$$

$$\text{whence } x = \tfrac{1}{4}.$$

CASE 3. *When the last term only is a square, or when the expression is of the form*

$$\sqrt{ax^4 + bx^3 + cx^2 + dx + e^2}.$$

Page 273.

Ex. 2. Here, $m = -\frac{9}{8}$, and $n = -\frac{3}{2}$.

Putting, then,

$$2x^4 - 3x + 1 = (-\frac{9}{8}x^2 - \frac{3}{2}x + 1)^2 = \frac{81}{64}x^4 + \frac{27}{8}x^3 - 3x + 1;$$

$$\text{we have } 2x^4 = \frac{81}{64}x^4 + \frac{27}{8}x^3;$$

$$\text{or, } 128x^4 = 81x^4 + 216x^3;$$

$$\text{whence } x = \frac{216}{47}.$$

Ex. 3. Here $m = -13$, and $n = 8$, therefore

$$\text{put } 22x^4 - 40x^3 - 40x^2 + 64x + 16 = (-13x^2 + 8x + 4)^2 =$$

$$169x^4 - 208x^3 - 40x^2 + 64x + 16;$$

$$\text{hence we have } 22x^4 - 40x^3 = 169x^4 - 208x^3;$$

$$147x = 168;$$

$$\therefore x = \frac{168}{147} = \frac{8}{7}.$$

Additional Examples.

Page 275.

Ex. 2. Put $x = y + 1$, and we shall have

$$x^4 - 2x^2 + 2 = y^4 + 4y^3 + 4y^2 + 1,$$

which must be a square; then, by the last case, let us represent this

square by

$$(2y^2+1)^2 = 4y^4+4y^2+1;$$

$$\text{hence } y^4+4y^3 = 4y^4, \text{ or } 3y = 4;$$

$$\therefore\ y = \tfrac{4}{3};$$

$$\text{consequently, } x = \tfrac{7}{3}.$$

Ex. 3. Put here $x = y+1$, and we shall have

$$22x^4-128x^3+212x^2-64x-26 = 22y^4-40y^3-40y^2+64y+16,$$

which must be a square; therefore, by the last case, denote this square by

$$(-13y^2+8y+4)^2 = 169y^4-208y^3-40y^2+64y+16,$$

$$\text{and we shall have } 22y^4-40y^3 = 169y^4-208y^3;$$

$$\therefore\ y = \frac{8}{7};$$

$$\text{whence } x = \frac{15}{7}.$$

PROBLEM IV.

To find such values of x as will render rational the expression

$$\sqrt[3]{ax^3+bx^2+cx+d}.$$

CASE I. *When both first and last terms are cubes, or when the expression is of the form*

$$\sqrt[3]{a^3x^3+bx^2+cx+d^3}.$$

Page 276.

Ex. 2. Put $-125x^3+89x^2+28x+8 = (-5x+2)^3 =$

$$-125x^3+150x^2-60x+8;$$

which gives $89x^2 + 28x = 150x^2 - 60x$;

$$\text{whence } x = \frac{88}{61}.$$

Ex. 3. Put $8x^3 + 42x^2 - 8x + 27 = (2x + 3)^3 = 8x^3 + 36x^2 + 54x + 27$;

and we shall have $42x^2 - 8x = 36x^2 + 54x$;

$$\therefore\ x = 10\tfrac{1}{3}.$$

Case 2. *When the first term only is a cube, or when the expression is of the form*

$$\sqrt[3]{a^3x^3 + bx^2 + cx + d}.$$

Page 277.

Ex. 2. Put $x^3 - 3x^2 + x = (x - 1)^3 = x^3 - 3x^2 + 3x - 1$;

and we shall obtain $x = 3x - 1$;

$$\therefore\ x = \tfrac{1}{2}.$$

This is the value resulting from the method given; but another solution may be easily obtained by equating all the terms, after the first, to zero, thus:

$$-3x^2 + x = 0,\ \therefore -3x + 1 = 0.$$

$$\text{whence } x = \tfrac{1}{3};$$

which is the answer given in the book.

Ex. 3. Put $x^3 + 3x^2 + 133 = (x + 1)^3 = x^3 + 3x^2 + 3x + 1$;

consequently, we have $3x + 1 = 133$;

$$\therefore\ x = 44.$$

CASE 3. *When the last term only is a cube, or when the expression is of the form*

$$\sqrt[3]{ax^3 + bx^2 + cx + d^3}.$$

Page 278.

Ex. 2. Put $3x^3 + 2x + 1 = (\frac{2}{3}x + 1)^3 = \frac{8}{27}x^3 + \frac{4}{3}x^2 + 2x + 1$;

whence we have $3x^3 = \frac{8}{27}x^3 + \frac{4}{3}x^2$;

$$\text{or } 81x^3 = 8x^3 + 36x^2;$$

$$\therefore\ x = \frac{36}{73}.$$

Ex. 3. Put $3x^3 - 6x^2 + 6x + 1 = (2x + 1)^3 = 8x^3 + 12x^2 + 6x + 1$;

hence $3x^3 - 6x^2 = 8x^3 + 12x^2$.

Consequently, $x = -\frac{18}{5}$.

Additional Examples.

Page 279.

Ex. 2. Put $x = y - 1$; and the expression then becomes

$$y^2 - y + 1;$$

which put equal to

$$(-\frac{1}{3}y + 1)^3 = -\frac{1}{27}y^3 + \frac{3}{9}y^2 - y + 1;$$

and we have

$$y^2 = -\frac{1}{27}y^3 + \frac{3}{9}y^2;$$

$$\text{or } 27y^2 = -y^3 + 9y^2,$$

$$\therefore y = -18;$$

$$\text{and } x = -19.$$

Ex. 3. Put here, $x = y + 1$; and the expression becomes

$$2y^3 + 6y^2 + 6y + 1,$$

which equate with

$$(2y + 1)^3 = 8y^3 + 12y^2 + 6y + 1;$$

and there results

$$2y^3 + 6y^2 = 8y^3 + 12y^2;$$

$$\text{whence } y = -1;$$

$$\text{consequently, } x = 0.$$

Hence, there are no other values besides 1; it being understood that, in diophantine questions, the value *zero* for the number sought is always disregarded.

MISCELLANEOUS DIOPHANTINE QUESTIONS.

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Question 13. Let x be one number, and $x + 1$ the other; then we shall only have to make

$$2x + 1 = \square = s^2;$$

$$\therefore x = \frac{s^2 - 1}{2};$$

where s may be any value whatever; but to get the least values of x

and $x + 1$ in whole numbers, we must evidently make $s = 3$, which gives for the numbers sought, 4 and 5.

Quest. 14. Let the two numbers be x^2 and y^2; then by the question

$$x^2 + y = \square;$$
$$y^2 + x = \square.$$

Assume

$$\sqrt{x^2 + y} = p - x;$$

and we shall obtain

$$x^2 + y = p^2 - 2px + x^2;$$

$$\text{whence } x = \frac{p^2 - y}{2p}.$$

Then, by substitution in the second equation, we have

$$y^2 + \frac{p^2 - y}{2p} = \square,$$

$$\text{that is, } y^2 - \frac{1}{2p}y + \frac{p}{2} = \square.$$

In order that this may be a square, it is necessary that the third term be equal to the square of half the coefficient of the second term, that is,

$\frac{p}{2} = \frac{1}{16p^2}$ whence

$$p^3 = \tfrac{1}{8}, \text{ or } p = \tfrac{1}{2};$$

$$\therefore x = \frac{p^2 - y}{2p} = \tfrac{1}{4} - y.$$

Consequently, any value less than $\frac{1}{4}$ taken for y will answer the conditions of the problem. If y be assumed equal to $\frac{1}{5}$, x will be $\frac{1}{20}$. If $y = \frac{1}{6}$, $x = \frac{1}{12}$, and these latter being squared, give $\frac{1}{36}$ and $\frac{1}{144}$, which are the numbers required.

Quest. 15. Let x and y be the numbers sought; then, by the question, we have to solve the equations

$$x^2 + xy, \text{ or } x(x+y) = m^2,$$
$$y^2 + xy, \text{ or } y(x+y) = n^2.$$

Assume $x = p^2$, and $y = q^2$, we shall then have

$$p^2(p^2+q^2) = m^2,$$
$$q^2(p^2+q^2) = n^2,$$

which conditions will be fulfilled, if we find $p^2 + q^2 =$ a square; now, for this purpose, let us put

$$p = r^2 - s^2 \text{ and } q = 2rs;$$
$$\text{hence, } x = p^2 = (r^2 - s^2)^2;$$
$$\text{and } y = q^2 = 4r^2s^2;$$

where r and s may be assumed at pleasure.

If $r = 2$, and $s = 1$, then $x = 9$, and $y = 16$. These, however, are square numbers, which is not a necessary condition.

It is obvious, that any multiple whatever of the same numbers will equally answer the conditions of the question.

We shall have, therefore, a more general solution by taking $x = t(r^2 - s^2)^2$, and $y = 4tr^2s^2$; where r, s, and t, may be any numbers whatever.

Quest. 16. Let x and y be the fractions sought; then, by the conditions of the question,

$$x^2 + y = y^2 + x,$$
$$\text{and } x^2 + y^2 = \square;$$

by transposing the first equation, and completing the square,

$$x^2 - x + \tfrac{1}{4} = y^2 - y + \tfrac{1}{4};$$

by solving which, we obtain

$$x = y, \text{ or } x = 1 - y;$$

substituting the latter of these values of x in the second equation, it becomes

$$1 - 2y + 2y^2 = \square.$$

Put $ry - 1$ for the side of this square, and we shall have

$$r^2y^2 - 2ry + 1 = 2y^2 - 2y + 1;$$

whence we obtain

$$y = \frac{2r - 2}{r^2 - 2},$$

$$\text{and } x = 1 - y = \frac{r^2 - 2r}{r^2 - 2};$$

when $r = 3$, $x = \frac{3}{7}$, and $y = \frac{4}{7}$, the fractions required.

Quest. 17. Let x and y be the two numbers sought; then

$$x^2 + y^2 + xy = \square.$$

$$\text{Assume } x^2 + xy + y^2 = (x + r)^2 = x^2 + 2rx + r^2,$$

and we shall have

$$x = \frac{y^2 - r^2}{2r - y};$$

where r and y may be taken at pleasure, provided that y be greater than r, but less than $2r$.

$$\text{If } y = 3, \text{ and } r = 2, \text{ then } x = 5;$$

$$y = 5, \text{ and } r = 3, \text{ then } x = 16;$$

$$y = 7, \text{ and } r = 6, \text{ then } x = \tfrac{13}{5}.$$

Quest. 18. If x, y, and z denote the numbers, then

$$x^2 + yz = \square, \; y^2 + xz = \square, \; z^2 + xy = \square.$$

Assume $x = mz$, $y = nz$, then the expressions become

$$m^2 + n = \square, \; n^2 + m = \square, \; 1 + mn = \square;$$

now, the first and second expressions are evidently squares, if

$$m + n = \tfrac{1}{4}, \therefore m = \tfrac{1}{4} - n;$$

by substitution,

$$1 + mn = 1 + \tfrac{1}{4}n - n^2 = \square = (1 - cn)^2;$$

$$\text{whence } n = \frac{2c + \frac{1}{4}}{c^2 + 1} = \frac{8c + 1}{4(c^2 + 1)}, \text{ and } m = \frac{1}{4} - n = \frac{c(c - 8)}{4(c^2 + 1)},$$

$$\therefore x = mz = \frac{c(c - 8)}{4(c^2 + 1)} z, \; y = nz = \frac{8c + 1}{4(c^2 + 1)} z;$$

hence, if z be taken $= 4(c^2 + 1)$, then

$$x = c(c - 8), \; y = 8c + 1;$$

where c may be any number greater than 8; taking $c = 9$, we obtain

$$z = 328, \; y = 73, \; x = 9.$$

Quest. 19. Let ax and bx denote the numbers; then $(a+b)x = \square = n^2$,

$$a^2 + b^2 = \square, \; (a^3 + b^3)x = \square,$$

$$\text{whence } x = \frac{n^2}{a + b};$$

and substituting this value of x in the third formula, we have

$$\frac{a^3 + b^3}{a + b} n^2 = \square,$$

$$\text{or } (a^2 - ab + b^2) n^2 = \square,$$

$$\text{or } a^2 - ab + b^2 = \square.$$

Put

$$\left.\begin{aligned} a^2 + b^2 &= A^2 \\ a^2 - ab + b^2 &= B^2 \end{aligned}\right\};$$

by subtraction,

$$ab = A^2 - B^2;$$

$$\text{now, take } A + B = 2a, \; A - B = \tfrac{1}{2}b,$$

$$\text{whence we obtain } A = a + \tfrac{1}{4}b,$$

$$\text{and } a^2 + b^2 = A^2 = a^2 + \tfrac{1}{2}ab + \tfrac{1}{16}b^2,$$

$$\text{or } 16b = 8a + b, \text{ or } 15b = 8a;$$

$$\text{hence } a = 15,\ b = 8;$$

$$\therefore\ x = \frac{n^2}{a+b} = \frac{n^2}{23} = 23, \text{ (when } n = 23);$$

Consequently, $ax = 345$, $bx = 184$, the numbers required.

Quest. 20. Let x, y, and z denote the numbers sought; then by the question

$$xyz + 1 = \square = v^2, \quad zy + 1 = \square = m^2,$$

$$zx + 1 = \square = n^2,\ xy + 1 = \square = r^2, \text{ or } xy = \frac{v^2-1}{z}:$$

$$\text{hence, } r^2 - 1 = \frac{v^2-1}{z}, \text{ or } z = \frac{v^2-1}{r^2-1};$$

and, by substitution,

$$\frac{v^2x - x}{r^2-1} + 1 = n^2, \text{ or } x = \frac{r^2n^2 - n^2 + 1 - r^2}{v^2-1};$$

and again, by substitution,

$$\frac{v^2y - y}{r^2-1} + 1 = m^2, \text{ or } y = \frac{m^2r^2 - m^2 + 1 - r^2}{v^2-1};$$

whence, as v, r, m, and n may be taken at pleasure, by making $m = 2$, $v = 5$, $r = 5$, and $n = 3$, we have $z = 1$, $x = 3$, and $y = 8$, which three values of x, y, and z, will answer the conditions of the question.

Quest. 21. Put x, y, z, for the three numbers sought; then, by the question,

$$x^2 + y = \square = (p - x)^2;$$

$$y^2 + z = \square = (q - y)^2;$$

$$z^2 + x = \square = (r - z)^2.$$

From these three equations, we obtain

$$x = \frac{r^2 - 2q^2r + 4p^2qr}{1 + 8pqr},\ y = \frac{p^2 - 2pr^2 + 4pq^2r}{1 + 8pqr},\ z = \frac{q^2 - 2p^2q + 4pqr^2}{1 + 8\,pqr};$$

whence

$$x + y + z =$$

$$\frac{\{q^2 - 2p^2q + p^2 + (4pq^2 - 2q^2 + 4p^2q)r + (4pq - 2p + 1)r^2\} \times (1 + 8pqr)}{(1 + 8pqr)^2}$$

which must be a square. It is evident the first term of the numerator will be a square when $p = 1$, and that we may not have any powers of r greater than the second, we shall make the coefficient of r^2 in the first factor equal to nothing; this will give us $q = \frac{1}{4}$. By substituting these values of p and q, and reducing, the formula to be made a square becomes

$$1 + 4r + 4r^2,$$

which is already the square of $1 + 2r$, so that the value of r may be taken at pleasure. If we take $r = \frac{1}{4}$, we shall find

$$x = \frac{39}{80}, y = \frac{1}{40}, z = \frac{1}{20},$$

which are numbers that will answer the conditions of the problem; and by changing the value of r, we may find other values of x, y, and z.

Or thus:

By the question, $x^2 + y$, $y^2 + z$, $z^2 + x$, $x + y + z$, must all be squares.

To make the first expression a square, assume $y = 1 - 2x$; then, the second and fourth expressions become

$$\left.\begin{array}{l} 1 - 4x + 4x^2 + z = \square = A^2 \\ 1 - x \qquad\quad + z = \square = B^2 \end{array}\right\},$$

$$\therefore\ A^2 - B^2 = 4x^2 - 3x.$$

Take $A + B = 2x$, $A - B = 2x - \frac{3}{2}$, from which we obtain

$$B = \tfrac{3}{4}:$$

$$\text{hence, } 1 - x + z = B^2 = \tfrac{9}{16},$$

$$\text{or } x = \tfrac{7}{16} + z;$$

by substitution,

$$z^2 + x = z^2 + z + \tfrac{7}{16} = \square,$$

$$\text{or } 16z^2 + 16z + 7 = (4z - \frac{p}{q})^2;$$

$$\text{whence } z = \frac{p^2 - 7q^2}{16q^2 + 8pq} = \frac{p^2 - 7q^2}{8(pq + 2q^2)};$$

$$\therefore\ x = \frac{7}{16} + z = \frac{p}{16q} \cdot \frac{2p + 7q}{p + 2q},\ y = 1 - 2x = \frac{16q^2 + pq - 2p^2}{8\,(pq + 2q^2)}.$$

If $p = 3$, $q = 1$, then the numbers are $\frac{39}{80}, \frac{1}{40}, \frac{1}{20}$.

If $p = 8$, $q = 3$, ... $\frac{37}{84}, \frac{5}{42}, \frac{1}{336}$.

If $p = 11$, $q = 4$, ... $\frac{275}{608}, \frac{29}{304}, \frac{9}{608}$.

Quest. 22. Let $\frac{1}{2}x^2 - y$, $\frac{1}{2}x^2$, and $\frac{1}{2}x^2 + y$, be the three numbers in arithmetical progression;

Then we have to find x^2, $x^2 + y$, and $x^2 - y$ rational squares; or

$$x^2 + y \text{ and } x^2 - y = \text{squares}.$$

Assume $y = 2rx + r^2$; then we shall have

$$x^2 + y = x^2 + 2rx + r^2 = (x + r)^2,$$

and, therefore, it only remains to make $x^2 - y$, or

$$x^2 - 2rx - r^2 = \square;$$

$$\text{put } x^2 - 2rx - r^2 = (x - m)^2 = x^2 - 2mx + m^2, \text{ then}$$

$$x = \frac{m^2 + r^2}{2m - 2r},$$

where m and r may be taken at pleasure.

If $m = 5$, and $r = 4$; then $x = \frac{41}{2}$, and $\frac{1}{2}x^2 = \frac{1681}{8}$. Also

$y = 2rx + r^2 = 41 \times 4 + 16 = 180$; whence the three numbers will be

$$30\tfrac{1}{8},\ 210\tfrac{1}{8},\ \text{and}\ 390\tfrac{1}{8};$$

and if these numbers be multiplied by any square number, the same conditions will obviously obtain; hence, multiplying by 4, we shall have

$$120\tfrac{1}{2},\ 840\tfrac{1}{2},\ \text{and}\ 1560\tfrac{1}{2},$$

which answer the conditions of the question.

Quest. 23. Let ax^2, ay^2, and $\frac{ay^4}{x^2}$, represent the three numbers, in geometrical progression: then, by the question,

$$(y^2 - x^2)\,a = \square,$$

$$\left(\frac{y^4 - x^4}{x^2}\right) a = \square,$$

$$\left(\frac{y^4 - x^2y^2}{x^2}\right) a = \frac{y^2}{x^2}\,(y^2 - x^2)\,a = \square.$$

Hence, making $y^2 - x^2 = am^2$, and $y^4 - x^4 = an^2$, all the conditions will be satisfied. Whence

$$y^2 + x^2 = \frac{n^2}{m^2};$$

Assume, therefore, $x = p^2 - q^2$, and $y = 2pq$; where p and q may be taken at pleasure. If $p = 2$, and $q = 1$, then $x = 3$, and $y = 4$; and the numbers are

$$9a,\ 16a,\ \text{and}\ \frac{256a}{9},$$

$$\text{or } 81a,\ 144a,\ \text{and}\ 256a,$$

a being $= \frac{y^2 - x^2}{m^2}$; when m^2 may be any square factor whatever of $y^2 - x^2$. In the present instance, let $m = 1$, then $a = 7$, and the required numbers are

567, 1008, and 1792.

Quest. 24. Let x^2, y^2, and z^2 denote the three squares. Then,

$$x^2 + y^2 = \square = a^2,\ x^2 + z^2 = \square = b^2,\ y^2 + z^2 = \square = c^2.$$

From these three equations we obtain x^2, y^2, and z^2, viz.:

$$x^2 = \tfrac{1}{2}(a^2 + b^2 - c^2),$$
$$y^2 = \tfrac{1}{2}(a^2 + c^2 - b^2),$$
$$z^2 = \tfrac{1}{2}(b^2 + c^2 - a^2);$$

or, by putting $b = m + n$, $c = m - n$, the expressions become

$$2a^2 + 8mn = \square,\ 2a^2 - 8mn = \square,\ 4m^2 + 4n^2 - 2a^2 = \square.$$

Assume $2a^2 = m^2 + 16n^2$, then the first and second expressions are squares, and, by substitution, the third becomes

$$3m^2 - 12n^2 = \square, \text{ also } 2m^2 + 32n^2 = 4a^2 = \square;$$
$$\text{or if } m = np,\ 3p^2 - 12 = \square,\ 2p^2 + 32 = \square.$$

Put the first of these expressions, viz.:

$$3p^2 - 12 = 3(p+2)(p-2) = f^2(p-2)^2;$$
$$\text{then } 3p + 6 = pf^2 - 2f^2,\ \therefore p = 2 \times \frac{f^2 + 3}{f^2 - 3}:$$

hence, by substitution, in the second expression, we have

$$8 \times \frac{(f^2+3)^2}{(f^2-3)^2} + 32 = \square,$$
$$\text{or } 10f^4 - 36f^2 + 90 = \square;$$

which is the case when $f = 1$. Put, then, $f = 1 + q$, and it becomes

$$64 - 32q + 24q^2 + 40q^3 + 10q^4 = \square = (8 - 2q + \tfrac{5}{4}q^2)^2 =$$
$$64 - 32q + 24q^2 - 5q^3 + \tfrac{25}{16}q^4;$$
$$\text{whence } 640 + 160q = -80 + 25q,$$
$$\text{or } 135q = -720,\ \therefore q = -\tfrac{16}{3},$$

$$\text{and } f = 1 + q = -\frac{13}{3},\ p = 2 \times \frac{f^2 + 3}{f^2 - 3} = \frac{196}{71}.$$

And since $m = np$, $\therefore m = \frac{196}{71} n$, where if $n = 71$, $m = 196$; consequently, $b = m + n = 267$, $c = m - n = 125$, and $a = 244$.

Substituting these values of a, b, and c, in the expressions for x^2, y^2, and z^2, we obtain

$$x^2 = \overline{240}|^2,\ y^2 = \overline{44}|^2, \text{ and } z^2 = \overline{117}|^2,$$

which appear to be the least numbers that the problem admits of.

Quest. 25. By the question, $x^2 - y^2$, $x^2 - z^2$, and $y^2 - z^2$ are to be squares; or $\frac{x^2}{z^2} - \frac{y^2}{z^2}$, $\frac{x^2}{z^2} - 1$, $\frac{y^2}{z^2} - 1$, dividing by z^2. Assume, now, $\frac{x}{z} = \frac{p^2+1}{p^2-1}$, $\frac{y}{z} = \frac{q^2+1}{q^2-1}$, which fulfils the second and third conditions, and, by substitution, the first becomes

$$\left(\frac{p^2+1}{p^2-1} + \frac{q^2+1}{q^2-1}\right)\left(\frac{p^2+1}{p^2-1} - \frac{q^2+1}{q^2-1}\right),$$

$$\text{or } \frac{4(p^2q^2-1)(q^2-p^2)}{(p^2-1)^2(q^2-1)^2},$$

and, since the denominator is a square, it only remains to make

$$(p^2q^2 - 1)(q^2 - p^2) = \square,$$

$$\text{or } (p^2q^2 - 1)\left(\frac{q^2}{p^2} - 1\right) = \square;$$

this is done by making

$$pq = \frac{a^2 + b^2}{2ab},\quad \frac{q}{p} = \frac{c^2 + d^2}{2cd},$$

by which each factor becomes a square; and since

$$pq \times \frac{q}{p} = q^2 = \frac{a^2 + b^2}{2ab} \times \frac{c^2 + d^2}{2cd},$$

therefore the product of these fractions must be a square; or

$$ab\,(a^2 + b^2)\,cd\,(c^2 + d^2) = \square.$$

$$\text{Put } a = f + g,\ b = f - g,\ c = h + k,\ d = h - k;$$

$$\text{then } 2\,(f^4 - g^4) \times 2\,(h^4 - k^4), \text{ or } (f^4 - g^4)\,(h^4 - k^4) = \square,$$

$$\text{or if } f = f'g,\ h = h'k,\ (f'^4 - 1)\,(h'^4 - 1) = \square,$$

rejecting the squares in each expression.

$$\text{Put } (f'^4 - 1)\,(h'^4 - 1) = (f'^4 - 1)^2\,(h'^2 - 1)^2;$$

$$\text{whence we have } 2h'^2 = f'^4 h'^2 - f'^4,$$

$$\text{or } h'^2 = \frac{f'^4}{f'^4 - 2},\ \ \therefore f'^4 - 2 = \square;$$

which is the case if

$$f'^2 = \frac{r^2 + 2s^2}{2rs},$$

and $r^2 + 2s^2$ is a square if

$$r = t^2 - 2,\ s = 2t:$$

$$\text{hence } f'^2 = \frac{(t^2 + 2)^2}{4t(t^2 - 2)},\ \ \therefore 4t(t^2 - 2) \text{ or } t(t^2 - 2) = \square,$$

which is evidently the case when $t = 2$;* therefore,

$$f' = \frac{3}{2},\ \ k' = \frac{9}{7},\ \ \text{and } f = \frac{3}{2}g,\ \ h = \frac{9}{7}k;$$

$$\therefore f = 3,\ g = 2,\ h = 9,\ k = 7,\ a = f + g = 5,\ b = f - g = 1,$$

$$c = h + k = 16,\ d = h - k = 2;$$

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* Other values of t may be found by the usual rules, viz. $t = \frac{9}{4}$, 338, &c.

whence $pq=\frac{13}{5}, \frac{q}{p}=\frac{65}{15}$, $\therefore q^2=\frac{\overline{13}|^2}{16}$, or $q=\frac{13}{4}$, and $p=\frac{4}{5}$;

by which means we have

$$\frac{x}{z}=-\frac{41}{9}, \frac{y}{z}=\frac{185}{153}, \text{ or } x=-\frac{41}{9}z, \ y=\frac{185}{153}z;$$

in order to obtain whole numbers, let $z=153$, then $x=-697$, and $y=185$; consequently the three squares are

$$\overline{153}|^2, \overline{185}|^2, \text{ and } \overline{697}|^2.$$

Or the formula $(p^2q^2-1)(\frac{q^2}{p^2}-1)$ may be made a square in a more general and satisfactory manner (though we cannot find such small numbers), by putting $\frac{q}{p}=m$, or $q=mp$, and it becomes

$$(m^2p^4-1)(m^2-1)=\square,$$

which is the case when $p=1$; let its root $=e$; then

$$e=m^2-1,$$

and put $m^2(m^2-1)=a$; whence $m^2e=a$, and, by substitution, $(m^2p^4-1)(m^2-1)=ap^4-e^2$, which is to be a square. Assume (*Art.* 171) $p=y+1$, then we shall have

$$a(y+1)^4-e^2, \text{ or } ay^4+4ay^3+6ay^2+4ay+e^2,$$

which must be a square. Compared with formula (*Art.* 170) we have

$$b=4a, \ c=6a, \ d=4a, \ e=m^2-1;$$

also

$$n'=\frac{4a}{2e}=\frac{4m^2(m^2-1)}{2(m^2-1)}=2m^2,$$

and

$$m'=\frac{6a}{2e}-\frac{4a^2}{2e^3}=3\frac{a}{e}-2\frac{a^2}{e^2e}=3m^2-2\frac{m^4}{e}=\frac{3m^2e-2m^4}{e}=\frac{m^2(m^2-3)}{e};$$

hence

$$y = \frac{2m^2 2m^2(m^2-3)-4m^2e}{m^2e - \frac{m^4(m^2-3)^2}{e^2}} = \frac{4m^3e(m^2-3)-4e^3}{e^3-m^2(m^2-3)^2} = \frac{4-4m^4}{3m^4-6m^2-1},$$

and

$$p = 1 + y = 1 + \frac{4-4\,m^4}{3m^4-6m^2-1} = \frac{6m^2+m^4-3}{6m^2-3m^4+1};$$

where m may be any number.

$$\text{Let } m = 2, \text{ then } p = -\frac{37}{23},\ q = -\frac{74}{23}:$$

$$\text{hence } \frac{x}{z} = \frac{949}{420}, \text{ and } \frac{y}{z} = \frac{6005}{4947}.$$

Quest. 26. Let x^2, $4x^2$, and $16x^2$ be the numbers assumed, in order to obtain the answer given; then

$$x^2 + x,\ 4x^2 + 2x,\ 4x^2 + x,$$

must be squares.

If we assume

$$x^2 + x = (px)^2 = p^2x^2, \text{ then } x = \frac{1}{p^2-1};$$

substituting this value of x in the second and third expressions, we have

$$2p^2 + 2 = \square,\ p^2 + 3 = \square;$$

which formulæ occur in the solution to Quest. 12; the value of p being $\frac{23}{7}$, we have

$$x = \frac{1}{p^2-1} = \frac{49}{480},$$

and hence the three squares are

$$\left(\frac{49}{480}\right)^2,\ \left(\frac{98}{480}\right)^2, \text{ and } \left(\frac{196}{480}\right)^2.$$

Quest. 27. Let x^2, y^2, and z^2 be the numbers; then, since they are to be in harmonical proportion, we must have (Art. 77.)

$$x^2 : z^2 :: x^2 - y^2 : y^2 - z^2;$$

$$\therefore x^2y^2 - x^2z^2 = x^2z^2 - y^2z^2;$$

$$\text{and } (x^2 + z^2)\, y^2 = 2x^2z^2;$$

$$\therefore y^2 = \frac{2x^2z^2}{x^2+z^2} = \frac{4x^2z^2}{2(x^2+z^2)};$$

and as the numerator of this fraction is a square, it remains to make

$$2\,(x^2 + z^2) = \square.$$

One case immediately presents itself, viz.: $x = 7$, $z = 1$, to which corresponds $y = \frac{7}{5}$, so that the following values fulfil the conditions of the question, viz.:

$$49,\ 1,\ \frac{49}{25};$$

and since any multiples of proportional quantities are also in proportion, we shall have, by multiplying each by 25, the following integral numbers, viz.

$$1225,\ 49,\ \text{and } 25.$$

which are the numbers required.

Quest. 28. By the question, $x + y = x^3 + y^3$, or if $x + y = s$, and $xy = p$, we have, by Quest. 5, (p. 129 of Alg.,)

$$s = s^3 - 3sp,\ \therefore p = \frac{s^2 - 1}{3},$$

$$\text{and } x \text{ or } y = \tfrac{1}{2}s \pm \tfrac{1}{2}\sqrt{(s^2 - 4p)} = \tfrac{1}{2}s \pm \tfrac{1}{2}\sqrt{\left(s^2 - \frac{4s^2-4}{3}\right)}$$

$$= \tfrac{1}{2}s \pm \tfrac{1}{6}\sqrt{(12 - 3s^2)}:$$

$$\text{hence } 12 - 3s^2 = \square.$$

Put $12 - 3s^2 = 3\,(2 + s)\,(2 - s) = c^2\,(2 - s)^2$,

$$\text{whence } s = 2 \times \frac{c^2 - 3}{c^2 + 3};$$

take $c = 5$, then $s = \frac{11}{7}$, $x = \frac{11}{14} \pm \frac{5}{14} = \frac{8}{7}$ or $\frac{3}{7}$:

and $\frac{8}{7}$, $\frac{3}{7}$ numbers required; or if $c = 9$, then

$$s = \tfrac{13}{7},\ x = \tfrac{13}{14} \pm \tfrac{3}{14} = \tfrac{8}{7} \text{ or } \tfrac{5}{7};$$

and the numbers are $\frac{8}{7}$ and $\frac{5}{7}$.

Or, dividing each side by $x + y$, we have

$$1 = x^2 - xy + y^2, \text{ or } x^2 - xy = 1 - y^2;$$

$$\text{take } x - y = r\,(1 + y),\ x = s\,(1 - y),$$

from which we have

$$y = \frac{s - r}{r + s + 1} = \frac{n^2 - m^2}{m^2 + n^2 + mn},$$

$$x = \frac{2rs + s}{r + s + 1} = \frac{2mn + n^2}{m^2 + n^2 + mn},$$

where m and n may be taken at pleasure.

Quest. 29. To make $a^3 + b^3 + c^3 - 3$ a square, assume $a = n - b$, and it becomes

$$n^3 - 3n^2b + 3nb^2 + c^3 - 3;$$

and since n may be any number, let $n = 3$, and it is

$$9b^2 - 27b + 24 + c^3,$$

which is easily made a square. To obtain the answer given, c must be taken $= 2$.

Quest. 30. Let x^4 be the sum of the two numbers, and x^3 their difference; then the numbers will be expressed by

$$\frac{x^4 + x^3}{2} \text{ and } \frac{x^4 - x^3}{2},$$

the sum of the squares of which is

$$\frac{2x^8 + 2x^6}{4},$$

which, by the question, is to be a cube, or $4x^2 + 4$ a cube; which put $= c^3$, then will

$$\frac{c^3 - 4}{4} = x^2, \text{ or } \frac{c^3 - 4}{4} \text{ a square.}$$

$$\text{Let } c = 5; \text{ then } x^2 = \frac{121}{4}, \text{ and } x = \frac{11}{2}; \text{ whence}$$

$$\frac{x^4 + x^3}{2} \text{ and } \frac{x^4 - x^3}{2} \text{ will be found } = \frac{17303}{2.16} \text{ and } \frac{11979}{2.16} \text{ or}$$

$$\frac{34606}{2.2.16} \text{ and } \frac{23958}{2.2.16},$$

which, when squared, the denominators will evidently be cubes. Hence the numerators

34606 and 23958

may be taken for the required numbers, which will be found to answer every requisite of the question.

Or thus:

Let $x + y$ and x be the numbers; then by the second and third conditions,

$$y = \text{cube, and } 2x^2 + 2xy + y^2 = \text{cube};$$

put $x = yz$; then

$$2y^2z^2 + 2y^2z + y^2 = \text{cube};$$

or, since y is a cube, we may reject y^2, and then

$$2z^2 + 2z + 1 = \text{cube} = (1 + \tfrac{2}{3} z)^3 = 1 + 2z + \tfrac{4}{3} z^2 + \tfrac{8}{27} z^3;$$

$$\therefore 54 = 36 + 8z, \text{ and } z = \tfrac{9}{4};$$

$$\text{hence } x = \tfrac{9}{4}y,$$

and since y = cube, let $y = (2r)^3 = 8r^3$; then $x = 18r^3$;

and, by the first condition,

$$2x + y = 44r^3 = \square,$$

or dividing by $4r^2$,

$$11r = \square;$$

$$\text{whence } r = 11;$$

$$\text{consequently, } x = 18r^3 = 18 \times 1331 = 23958,$$

$$\text{and } x + y = 26r^3 = 26 \times 1331 = 34606.$$

Quest. 31. In order to give a general solution to this question, let a, b, c, and x be the four numbers; then a, b, and c may be any numbers assumed, so that

$$abc + 1 = \square;$$

by the other conditions of the question

$$abx + 1 \text{ or } mx + 1 = \square,$$

$$acx + 1 \text{ or } nx + 1 = \square,$$

$$bcx + 1 \text{ or } px + 1 = \square,$$

putting m, n, and p, for ab, ac, and bc, respectively.

To make $1 + mx$ a square, assume its root $= 1 - rx$, and we have

$$x = \frac{2r + m}{r^2};$$

substituting this value of x in the other two expressions,

$$\left.\begin{aligned} r^2 + 2rn + mn = \square = \text{A}^2 \\ r^2 + 2rp + mp = \square = \text{B}^2 \end{aligned}\right\};$$

by subtraction,

$$\text{A}^2 - \text{B}^2 = 2r(n - p) + m(n - p) = (2r + m)(n - p).$$

Now, take

$$\left.\begin{array}{l} \text{A} + \text{B} = 2r + m \\ \text{A} - \text{B} = \;\; n - p \end{array}\right\},$$

and we get $\text{A} = r + \frac{1}{2}(m + n - p)$;

hence $r^2 + 2rn + mn = \text{A}^2 = r^2 + r(m + n - p) + \frac{1}{4}(m + n - p)^2$:

from which we obtain

$$r = \frac{mn - \frac{1}{4}(m + n - p)^2}{m - n - p} = \frac{a^2bc - \frac{1}{4}(ab + ac - bc)^2}{ab - ac - bc}.$$

To obtain the answer given, take $a = \frac{1}{2}$, $b = 2$, $c = 3$, and we shall have, by substituting these values in the value of r,

$$r = \frac{25}{16} \times \frac{2}{13} = \frac{25}{104};$$

hence $x = \dfrac{2r + m}{r^2} = \dfrac{2r + ab}{r^2} = \dfrac{154}{104} \times \dfrac{\overline{104}|^2}{\overline{25}|^2} = \dfrac{16016}{25}$:

and therefore the four numbers are

$$\frac{1}{2},\ 2,\ 3,\ \text{and}\ \frac{16016}{625};$$

and an infinity of other sets of numbers might be found by giving other values to a, b, and c, so that $abc + 1 = \square$.

FINIS.

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